Renormalization group in stochastic theory of developed turbulence 1

Kolmogorov scaling and the stochastic Navier-Stokes problem

Juha Honkonen

National Defence University, Helsinki, Finland

Outline

- Inertial range in developed turbulence
- Kolmogorov scaling of structure functions
- Stochastic hydrodynamics
- Thermal fluctuations vs. random stirring
- Long-range correlated random force
- Field-theoretic representation
- Feynman rules for perturbation theory
- Graphs for pair correlation function
- UV regularization by the falloff exponent
- UV divergences as poles in ε

Inertial range in developed turbulence

Fully developed turbulence: at large *Reynolds numbers*

$$\mathrm{Re} = \frac{UL}{\nu} \gg \mathrm{Re}_{\,\mathrm{cr}}$$

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Cascade picture: energy transfer by turbulent eddies from the *integral scale* L to the *dissipation scale* η.

energy injection	cascade transfer	energy dissipation at mean rate $\overline{\varepsilon}$
$L = \frac{1}{m}$	$m \ll k \ll \Lambda$	$\eta = \frac{1}{\Lambda}$
energy range	*inertial range*	*dissipation range*

Kolmogorov scaling of structure functions

Statistical description of the turbulent flow by *structure functions* of the velocity field

$$S_n(r) = \left\langle \left[v_{\parallel}(t, \mathbf{x} + \mathbf{r}) - v_{\parallel}(t, \mathbf{x}) \right]^n \right\rangle \,, \qquad v_{\parallel} = \frac{\mathbf{v} \cdot \mathbf{r}}{r} \,.$$

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Kolmogorov constant C_K and $\frac{4}{5}$ (at $d=3$) law

$$S_2(r) \sim C_K (\overline{\varepsilon}\, r)^{2/3} , \quad S_3(r) \sim -\frac{12}{d(d+2)} \overline{\varepsilon}\, r .$$

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Anomalous scaling: exponents of S_n nonlinear in n.

Stochastic hydrodynamics

Fluctuating velocity generated by randomly forced Navier-Stokes equation for incompressible fluid ($\nabla \cdot \mathbf{v} = 0$)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - \frac{\nabla p}{\rho} + \mathbf{f}\,.$$

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Gaussian distribution of random force $\mathbf{f}$: $\langle\, f_m(t, \mathbf{k})\,\rangle = 0$ and

$$\langle\, f_m(t, \mathbf{k}) f_n(t', \mathbf{k}')\,\rangle = \left(\delta_{mn} - \frac{k_m k_n}{k^2}\right)(2\pi)^d \delta(t - t')\delta(\mathbf{k} + \mathbf{k}')\, d_f(k)\,.$$

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Force correlations yield mean energy injection rate $\overline{\mathcal{E}}$:

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k} d_f(k)\,.$$

The choice of the function $d_f(k)$ fixes perturbation theory.

Thermal fluctuations vs. random stirring

Near-equilibrium thermal fluctuations described by

$$d_f(k) = D_{20} k^2 \theta(\Lambda - k)\,, \qquad D_{20} = 2\nu_0 T/\rho\,. \qquad \text{(model A)}$$

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Calculations with cutoff tough. Without cutoff – UV injection.

Long-range correlated random force

Technically a more flexible choice is (ε is a new parameter)

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$$\delta(\mathbf{k}) = S_d^{-1} k^{-d} \lim_{\varepsilon \to 2} \left[(4-2\varepsilon)(k/\Lambda)^{4-2\varepsilon} \right], \quad S_d \equiv 2\pi^{d/2}/\Gamma(d/2)\,.$$

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With a suitable choice of D_{10} the physical value $\varepsilon \to 2$ corresponds to idealized injection by infinitely large eddies:

$$d_f(k) = 2(2\pi)^d\, \overline{\mathcal{E}}\, \delta(\mathbf{k})/(d-1)\,.$$

Field-theoretic (MSR) representation

Cast the Navier-Stokes problem into the field-theoretic form: De Dominicis-Janssen (or Martin-Siggia-Rose) action

$$S_{\rm NS}(\mathbf{v},\mathbf{v}') = \frac{1}{2}\mathbf{v}' D\mathbf{v}' - \mathbf{v}'\left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu_0 \nabla^2 \mathbf{v}\right] \ ,$$

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where integrals and sums implied and ($P_{mn} = \delta_{nm} - k_n k_m / k^2$)

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The MSR action generates perturbation expansion, e.g. through the integral representation of correlation functions

$$G_n(1,\ldots,n) = \langle v(1)\ldots v(n)\rangle = \int \mathcal{D}v \int \mathcal{D}v' v(1)\ldots v(n)\, e^{S_{\rm NS}(\mathbf{v},\mathbf{v}')}$$

Feynman rules for perturbation theory

Quadratic part of the MSR action gives rise to the bare propagator and correlation function depicted by lines

$$\langle v_m(\omega,\mathbf{k}) v'_n(-\omega,-\mathbf{k})\rangle_0 = \frac{P_{mn}}{-\mathrm{i}\omega + \nu_0 k^2} \Longleftrightarrow$$  ,

$$\langle v_m(\omega,\mathbf{k}) v_n(-\omega,-\mathbf{k})\rangle_0 = \frac{d_f(k) P_{mn}}{\omega^2 + \nu_0^2 k^4} \Longleftrightarrow$$,

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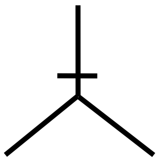
The interaction term $-v'(v\nabla)v = \varphi'_i V_{ijs}\varphi_j\varphi_s/2$ gives rise to the vertex factor ($\mathbf{k}$ is the wave-vector argument of the field v')

$$V_{ijs}(\mathbf{k}) = \mathrm{i}(k_j\delta_{is}+k_s\delta_{ij}) \Longleftrightarrow$$  .

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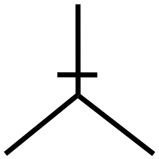
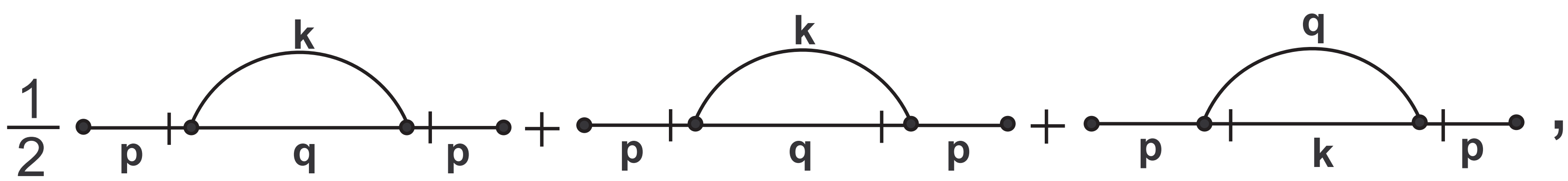
$$V_{ijs}(\mathbf{k}) = \mathrm{i}(k_j\delta_{is}+k_s\delta_{ij}) \Longleftrightarrow$$  .

The effective expansion parameter is $g_{10}\equiv D_{10}/\nu_0^3$.

Graphs for pair correlation function

One-loop graphs for the velocity-velocity correlation function:



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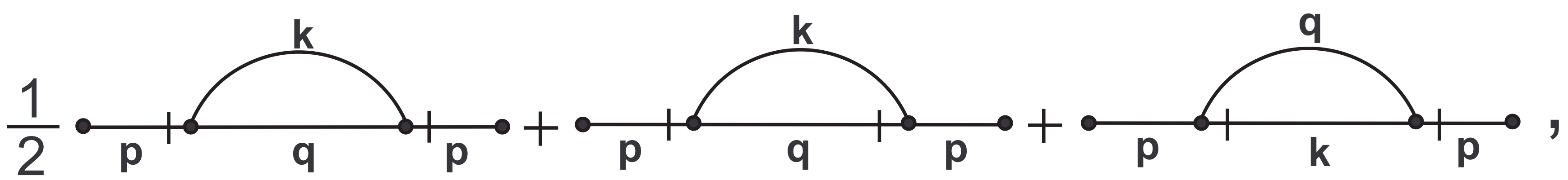

The second graph, e.g., corresponds to ($\mathbf{q} = \mathbf{p} - \mathbf{k}$)

$$R_2 = \frac{P_{ml}(\mathbf{p})}{-\mathrm{i}\omega + \nu_0 p^2} \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\Omega}{2\pi} V_{lij}(\mathbf{p}) \frac{P_{ii'}(\mathbf{q})}{-\mathrm{i}\,(\omega - \Omega) + \nu_0 q^2}$$

$$\times \frac{d_f(k) P_{jj'}(\mathbf{k})}{\omega^2 + \nu_0^2 k^4} V_{i'j'l'}(\mathbf{q}) \frac{d_f(p) P_{l'n}(\mathbf{p})}{\omega^2 + \nu_0^2 p^4} \,.$$

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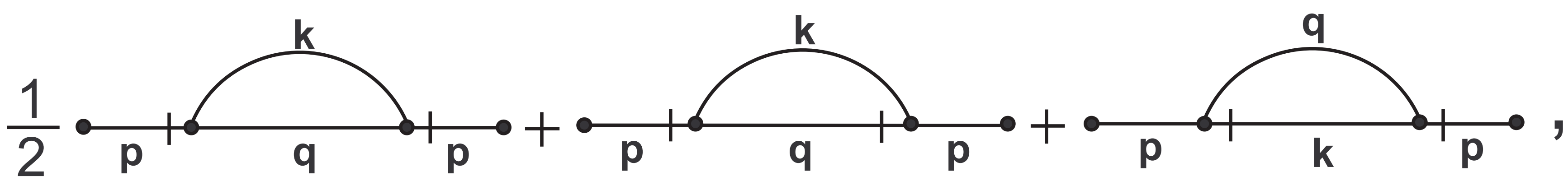

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For the powerlike force correlation $d_f(k) = D_{10}\, k^{4-d-2\varepsilon}$ the integral UV converges at $\varepsilon > 0$, diverges at $\varepsilon \leq 0$ ($\forall d$).

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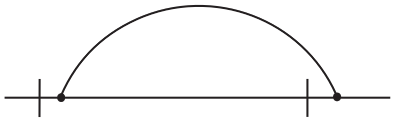
$\Gamma^{(1)} =$  .

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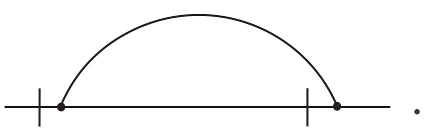

Frequency integration in the limit $p \to 0$ gives ($z \equiv (\mathbf{p}\cdot\mathbf{k})/pk$)

$$\left.\frac{\mathrm{Tr}\Gamma^{(1)}(\omega,\mathbf{p})}{\nu_0 p^2 (d-1)}\right|_{\substack{\omega=0\\ \mathbf{p}=0}} =$$

$$-\frac{g_0}{4(d-1)(2\pi)^d}\int \frac{d\mathbf{k}}{k^{d+2\varepsilon}}\, h(m/k)\left[d-3+(9-d)z^2-6z^4\right].$$

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$$\langle z^{2n} \rangle_\Omega = \frac{\Gamma(d/2)}{2\pi^{d/2}} \int d\Omega \cos^{2n}\theta = \frac{(2n-1)!!}{d(d+2)\ldots(d+2n-2)}, \ \langle z^{2n+1} \rangle_\Omega = 0.$$

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This gives ($\bar{S}_d = 2\pi^{d/2}/(2\pi)^d \Gamma(d/2)$, $g_{10} \equiv D_{10}/\nu_0^3$)

$$\Gamma^{(1)} = -\frac{g_0 \, \bar{S}_d \, (d-1)}{4(d+2)} \int_m^\infty \frac{dk}{k^{1+2\varepsilon}} = -\frac{(d-1)(m)^{-2\varepsilon} \, g_0 \bar{S}_d}{8(d+2)\,\varepsilon} \, .$$

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The pole explicit pole in ε will be absorbed in the UV renormalization of the 1PI response function with this graph.