Renormalization group in stochastic theory of developed turbulence

2. Renormalization and the renormalization group

Juha Honkonen

Outline

- Canonical dimensions of fields and parameters
- Canonical dimensions in hydrodynamics
- One-irreducible correlation functions
- Power counting for one-irreducible functions
- Renormalization and counterterms
- Renormalized correlation functions
- Homogeneous renormalization-group equation
- Invariant (running) parameters
- Critical dimensions at the fixed point of RG
- Scaling in terms of physical variables

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Total dimension of a quantity Q from the homogeneity of the free-field action under scaling $k \to \lambda k$, $\omega \to \lambda \omega$; here:

$$d_Q = d_Q^k + 2 d_Q^\omega \,.$$

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Dimensions of the viscosity and coupling constant for $d_f(k)=g_{10}\nu_0^3\,k^{4-d-2\varepsilon}$ (don't forget the Fourier transform!):

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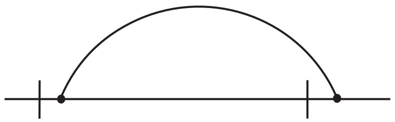
The theory is logarithmic at $\varepsilon = 0$ in any space dimension d.

One-irreducible correlation functions

Only one graph for the one-irreducible function $\langle vv' \rangle_{1-ir}$

$\Gamma^{(1)} =$  .

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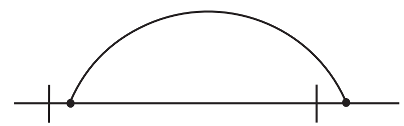
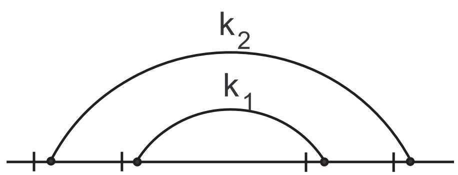
$\Gamma^{(1)} =$  .

There are, however, 8 graphs in the two-loop approximation

$\Gamma^{(2)}_1 =$ 

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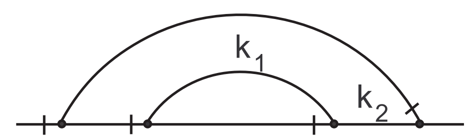
$\Gamma^{(2)}_2 =$ 

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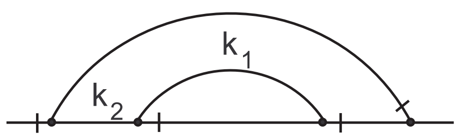
$\Gamma^{(2)}_3 =$ 

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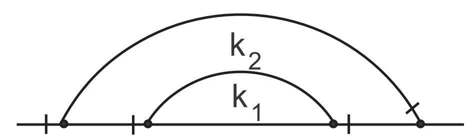
$\Gamma^{(2)}_4 = \frac{1}{2}$ 

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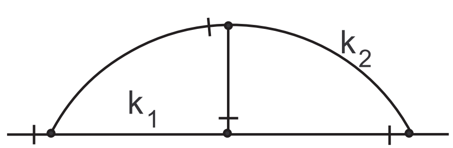
$\Gamma^{(2)}_5 =$ 

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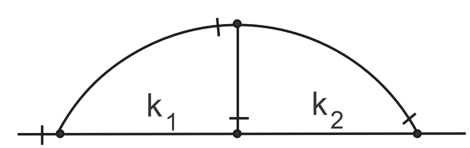
$\Gamma^{(2)}_6 =$ 

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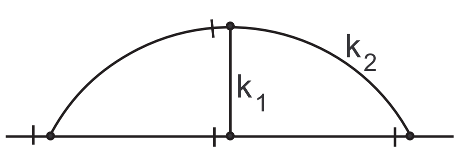
$\Gamma^{(2)}_7 =$ 

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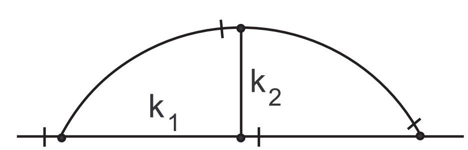
$\Gamma^{(2)}_8 =$ 

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The degree of divergence of a graph Γ is its dimension d_Γ minus the number of factorizing vertex wave vectors:

$$\delta_{\Gamma_{v^N v'^{N'}}} = d + 2 - V d_{g_0} - N d_v - N' d_{v'} - N' \,.$$

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The UV-divergent part of a graph is *subtracted* to produce a UV-finite *renormalized* function and a *counterterm*.

Renormalization and counterterms

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Separate analysis for $d = 2$, let $d > 2$ for the time being.

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Basic statement of renormalization theory: UV divergences of a renormalizable model may be absorbed in a *redifinition of parameters* such that the *renormalized action*

$$S_{\mathrm{NS}}(\mathbf{v},\mathbf{v}') = \frac{1}{2}\mathbf{v}' D\mathbf{v}' - \mathbf{v}'\left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right]$$

generates UV finite correlation functions such that

$$\int \mathcal{D}v\mathcal{D}v' v(1)\ldots v(n)\, e^{S_{\mathrm{NS}0}(\mathbf{v},\mathbf{v}')} = \int \mathcal{D}v\mathcal{D}v' v(1)\ldots v(n)\, e^{S_{\mathrm{NS}}(\mathbf{v},\mathbf{v}')}\,.$$

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Most straightforwardly the *renormalization constant* Z_ν is found just from this condition.

Homogeneous renormalization-group equation

Introduce a scaling parameter μ in the connection between renormalized and unrenormalized (bare) parameters:

$$\nu_0 = \nu Z_\nu, \qquad g_{01} = D_{01}\nu_0^{-3} = g_1 \mu^{2\epsilon} Z_\nu^{-3}\,.$$

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$$[\mu\partial_\mu + \beta_1\partial_{g_1} - \gamma_\nu \nu\partial_\nu]\, G = 0 , \quad \gamma_\nu = \mu\partial_\mu\big|_0 \ln Z_\nu , \quad \beta_1 = \mu\partial_\mu\big|_0 g_1$$

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where derivatives taken with bare parameters fixed and

$$\int d\mathbf{r} \, \exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r})\right] \langle v_n(t,\mathbf{x}+\mathbf{r}) v_m(t,\mathbf{x})\rangle = P_{nm}(\mathbf{k}) G(k)$$

Invariant (running) parameters

RG solution for the velocity correlation function:

$$G(k) = \nu^2 k^{2-d} R\left(\frac{k}{\mu}, g_1, \frac{m}{\mu}\right) = \bar{\nu}^2 k^{2-d} R\left(1, \bar{g}_1, \frac{m}{k}\right) .$$

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Connection between bare and invariant parameters:

$$g_{10} = \bar{g}_1 k^{2\varepsilon} Z_\nu^{-3}\left(\bar{g}_1, \frac{m}{k}\right) , \quad \bar{\nu} = \left(\frac{D_{10} k^{-2\varepsilon}}{\bar{g}_1}\right)^{1/3} .$$

Critical dimensions at the fixed point of RG

For $\varepsilon > 0$ $\exists$ an IR-stable fixed point: $\bar{g}_1 \to g_{1*} \propto \varepsilon$.
Asymptotics of correlation and response functions W are generalized homogeneous functions

$$W\big|_{IR}(\{\lambda^{-\Delta_\omega} t_i\}, \{\lambda^{-1}\mathbf{x}_i\}) = \lambda^{\sum_\Phi \Delta_\Phi} W\big|_{IR}(\{t_i\}, \{\mathbf{x}_i\})\,.$$

Here, Δ_ω and Δ_Φ are critical dimensions of ω and $\Phi = \{v, v'\}$.

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They are all expressed through $\gamma_\nu^* \equiv \gamma_\nu(g_*)$:

$$\Delta_v = 1 - \gamma_\nu^* \,, \quad \Delta_{v'} = d - \Delta_\varphi \,, \quad \Delta_\omega = 2 - \gamma_\nu^* \,.$$

Due to Galilei invariance, basic critical dimensions are exact:

$$\Delta_v = 1 - 2\varepsilon/3 \,, \quad \Delta_\omega = 2 - 2\varepsilon/3 \,.$$

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$$G(k) \sim (D_{10}/g_{1*})^{2/3}\, k^{\,2-d-4\varepsilon/3} R(1, g_{1*}, u)\,,\; R(1, g_{1*}, u) = \sum_{n=1}^{\infty} \varepsilon^n R_n(u)$$

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Translate in traditional variables; trade D_{10} for the mean energy injection rate $\overline{\mathcal{E}}$ ($2 > \varepsilon > 0$):

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\; \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

Freezing of dimensions in the inertial range

Large-scale scaling in terms of $\overline{\mathcal{E}}$ and ν_0 for $2 > \varepsilon > 0$:

$$G(k) \sim \left[4(2-\varepsilon)/\overline{S}_d(d-1)g_{1*}\right]^{2/3} \nu_0^{2-\varepsilon}\,\overline{\mathcal{E}}^{\,\varepsilon/3} k^{\,2-d-4\varepsilon/3} R(1, g_{1*}, u)\,.$$

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Analysis of the inertial-range limit $u = m/k \to 0$ beyond RG.