# Renormalization group in stochastic theory of developed turbulence 2

## *Renormalization and the renormalization group*

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# Outline

- Canonical dimensions of fields and parameters
- Canonical dimensions in hydrodynamics
- Canonical dimensions of Green functions
- One-irreducible correlation functions
- Power counting for one-irreducible functions
- Renormalization and counterterms
- Renormalized correlation functions
- Homogeneous renormalization-group equation
- Invariant (running) parameters
- Critical dimensions at the fixed point of RG
- Scaling in terms of physical variables

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Total dimension of a quantity $Q$ from the homogeneity of the free-field action under scaling $k \to \lambda k$, $\omega \to \lambda\omega$; here:

$$d_Q = d_Q^k + 2d_Q^\omega \,.$$

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The theory is logarithmic at $\varepsilon = 0$ in any space dimension $d$.

# Canonical dimensions of generating functions

The basic generating function is rendered dimensionless

$$G(\mathbf{J},\mathbf{J}') = \int \mathcal{D}v \int \mathcal{D}v'\, e^{S_{\rm NS}(\mathbf{v},\mathbf{v}')+\mathbf{vJ}+\mathbf{v}'\mathbf{J}'}$$

$$= e^{\frac{1}{2}\frac{\delta}{\delta\mathbf{v}}\Delta_{11}\frac{\delta}{\delta\mathbf{v}}+\frac{\delta}{\delta\mathbf{v}}\Delta_{12}\frac{\delta}{\delta\mathbf{v}'}}\, e^{\mathbf{v}'(\mathbf{v}\nabla)\mathbf{v}+\mathbf{vJ}+\mathbf{v}'\mathbf{J}'}\Big|_{\mathbf{v}=\mathbf{v}'=0}$$

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Generating functions $W(\mathbf{J},\mathbf{J}') = \ln G(\mathbf{J},\mathbf{J}')$ and

$$\Gamma(\mathbf{v},\mathbf{v}') = W(\mathbf{J},\mathbf{J}') - \mathbf{v}\mathbf{J} - \mathbf{v}'\mathbf{J}'\,, \quad \mathbf{v} = \frac{\delta W}{\delta \mathbf{J}}\,,\ \mathbf{v}' = \frac{\delta W}{\delta \mathbf{J}'}$$

have zero canonical dimensions by definition.

# Canonical dimensions of Green functions

Green functions are derivatives of generating functionals; hence, in the coordinate space

$$d_{W_{n,n'}(\{t_i\},\{\mathbf{x}_i\})} = d_{G_{n,n'}(\{t_i\},\{\mathbf{x}_i\})} = n d_v + n' d_{v'} \, ,$$

$$d_{\Gamma_{n,n'}(\{t_i\},\{\mathbf{x}_i\})} = n d_J + n' d'_J = (n+n')(d+2) - n d_v - n' d_{v'} \, .$$

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In Fourier transforms frequency and wave-vector $\delta$ functions of conservation laws factorize. Thus, in the Fourier space

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For instance

$$d_{\Gamma_{22}} = d + 2 - 2(d-1) = 4 - d\,, \quad d_{W_{12}} = d + 2 + d - 2(d+2) = -2\,.$$

# Power counting for one-irreducible functions

The superficial degree of divergence $\delta$ of a 1PI graph $\gamma(\{\omega_i\},\{\mathbf{k}_i\})$ is defined by the homogeneity relation

$$\gamma(\{\lambda^2\omega_i\},\{\lambda\mathbf{k}_i\}) = \lambda^\delta\gamma(\{\omega_i\},\{\mathbf{k}_i\})$$

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The real degree of divergence $\delta'$ of a graph $\gamma$ is obtained by subtracting minus *the number of factorizing vertex wave vectors*. In our case

$$\delta'_\gamma = d + 2 - V d_{g_0} - N d_v - N' d_{v'} - N' \,.$$

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- Infinite number of 1PI functions with graphs $\delta' \geq 0$: *nonrenormalizable model*.

# Divergent 1PI functions of the NS problem

UV divergences in the one-irreducible graphs $\Gamma$. Let $d_{g_0} = 0$.
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$$\delta_\gamma = d_\Gamma \, .$$

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Graphs with $N' = 0$ and $N' = 1$, $N = 0$ vanish: $\delta \geq 0$ for the one-irreducible correlation functions $\langle vv' \rangle_{1-ir}$, $\langle vvv' \rangle_{1-ir}$:

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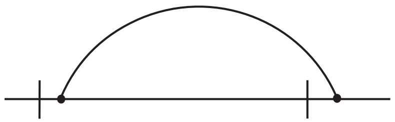
The UV-divergent part of a graph is *subtracted* to produce a UV-finite *renormalized* function and a *counterterm*.

# One-irreducible correlation functions

One-loop graph for the one-irreducible function $\langle vv' \rangle_{1-ir}$

$\Gamma^{(1)} =$  .

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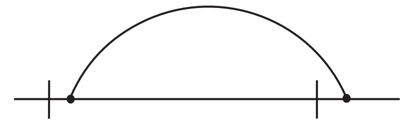
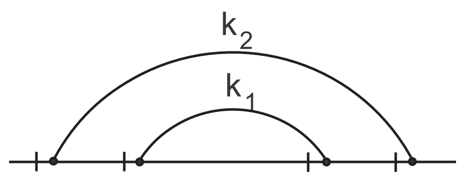
$\Gamma^{(1)} =$  .

There are, however, 8 graphs in the two-loop approximation

$\Gamma^{(2)}_1 =$ 

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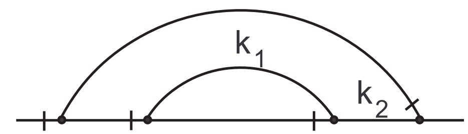
$\Gamma^{(2)}_2 =$ 

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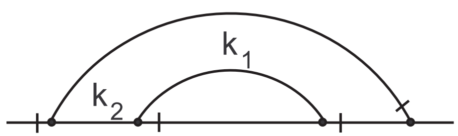
$\Gamma^{(2)}_3 =$ 

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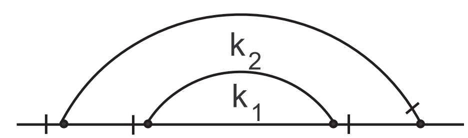
$\Gamma^{(2)}_4 = \frac{1}{2}$ 

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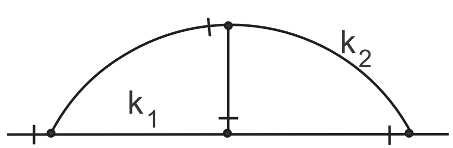
$\Gamma^{(2)}_5 =$ 

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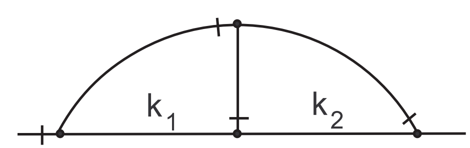
$\Gamma^{(2)}_6 =$ 

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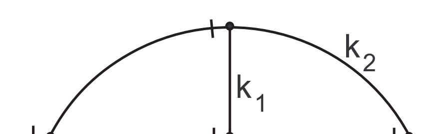
$\Gamma^{(2)}_7 =$ 

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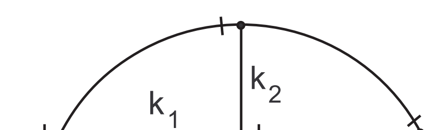
$\Gamma^{(2)}_8 =$ 

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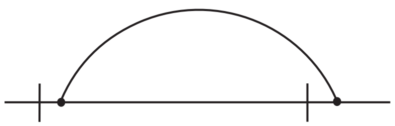
$\Gamma^{(1)} =$  .

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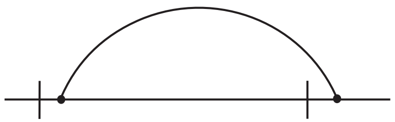
$\Gamma^{(1)} =$  .

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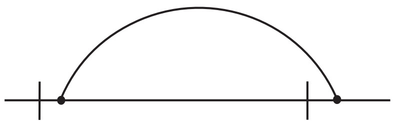
$\Gamma^{(1)} =$  .

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To subtract this term, add to the action the counterterm

$$\mathbf{v}' \nu_0 \frac{(d-1)^2 (m)^{-2\varepsilon}\, g_0 \bar{S}_d}{8(d+2)\,\varepsilon} \nabla^2 \mathbf{v} \,.$$

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Separate analysis for $d = 2$, let $d > 2$ for the time being.

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Basic statement of renormalization theory: UV divergences of a renormalizable model may be absorbed in a *redifinition of parameters* such that the *renormalized action*

$$S_{\mathrm{NSR}}(\mathbf{v},\mathbf{v}') = \frac{1}{2}\mathbf{v}' D\mathbf{v}' - \mathbf{v}' \left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right]$$

generates UV finite correlation functions such that

$$\int \mathcal{D}v\mathcal{D}v' v(1)\ldots v(n)\, e^{S_{\mathrm{NS}}(\mathbf{v},\mathbf{v}')} = \int \mathcal{D}v\mathcal{D}v' v(1)\ldots v(n)\, e^{S_{\mathrm{NSR}}(\mathbf{v},\mathbf{v}')}\,.$$

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Most straightforwardly the *renormalization constant* $Z_\nu$ is found just from this condition.

# Homogeneous renormalization-group equatio

Introduce a scaling parameter $\mu$ in the connection between renormalized and unrenormalized (bare) parameters:

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Bare quantities independent of $\mu$: the homogeneous RG equation; for the pair correlation function

$$[\mu\partial_\mu + \beta_1\partial_{g_1} - \gamma_\nu \nu\partial_\nu]\, G = 0 , \quad \gamma_\nu = \mu\partial_\mu\big|_0 \ln Z_\nu , \quad \beta_1 = \mu\partial_\mu\big|_0 g_1$$

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where derivatives taken with bare parameters fixed and

$$\int d\mathbf{r} \, \exp\left[\mathrm{i}(\mathbf{k}\cdot\mathbf{r})\right] \langle v_n(t,\mathbf{x}+\mathbf{r}) v_m(t,\mathbf{x})\rangle = P_{nm}(\mathbf{k}) G(k)$$

# Invariant (running) parameters

RG solution for the velocity correlation function:

$$G(k) = \nu^2 k^{2-d} R\left(\frac{k}{\mu}, g_1, \frac{m}{\mu}\right) = \bar{\nu}^2 k^{2-d} R\left(1, \bar{g}_1, \frac{m}{k}\right) \ .$$

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Invariant (running) parameters $\bar{g}_1(\mu/k, g_1)$, $\bar{\nu}(\mu/k, g_1)$: first integrals of the RG equation, e.g.

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Connection between bare and invariant parameters:

$$g_{10} = \bar{g}_1 k^{2\varepsilon} Z_\nu^{-3}\left(\bar{g}_1, \frac{m}{k}\right), \quad \bar{\nu} = \left(\frac{D_{10} k^{-2\varepsilon}}{\bar{g}_1}\right)^{1/3} .$$

# Critical dimensions at the fixed point of RG

For $\varepsilon > 0$ $\exists$ an IR-stable fixed point: $\bar{g}_1 \to g_{1*} \propto \varepsilon$.
Asymptotics of correlation and response functions $W$ are generalized homogeneous functions

$$W\big|_{IR}(\{\lambda^{-\Delta_\omega} t_i\}, \{\lambda^{-1}\mathbf{x}_i\}) = \lambda^{\sum_\Phi \Delta_\Phi} W\big|_{IR}(\{t_i\}, \{\mathbf{x}_i\})\,.$$

Here, $\Delta_\omega$ and $\Delta_\Phi$ are critical dimensions of $\omega$ and $\Phi = \{v, v'\}$.

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Here, $\Delta_\omega$ and $\Delta_\Phi$ are critical dimensions of $\omega$ and $\Phi = \{v, v'\}$.
They are all expressed through $\gamma_\nu^* \equiv \gamma_\nu(g_*)$:

$$\Delta_v = 1 - \gamma_\nu^* \, , \quad \Delta_{v'} = d - \Delta_\varphi \, , \quad \Delta_\omega = 2 - \gamma_\nu^* \, .$$

Due to Galilei invariance, basic critical dimensions are exact:

$$\Delta_v = 1 - 2\varepsilon/3 \, , \quad \Delta_\omega = 2 - 2\varepsilon/3 \, .$$

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IR fixed point yields large-scale limit ($k \to 0$, $u = m/k =$const)

$$G(k) \sim (D_{10}/g_{1*})^{2/3}\, k^{\,2-d-4\varepsilon/3} R(1, g_{1*}, u)\,,\; R(1, g_{1*}, u) = \sum_{n=1}^{\infty} \varepsilon^n R_n(u)$$

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Translate in traditional variables; trade $D_{10}$ for the mean energy injection rate $\overline{\mathcal{E}}$ ($2 > \varepsilon > 0$):

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k) \;\Rightarrow\; D_{10} = \frac{4(2-\varepsilon)\,\Lambda^{2\varepsilon-4}\overline{\mathcal{E}}}{\overline{S}_d(d-1)}\,,\; \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}\,.$$

# Freezing of dimensions in the inertial range

Large-scale scaling in terms of $\overline{\mathcal{E}}$ and $\nu_0$ for $2 > \varepsilon > 0$:

$$G(k) \sim \left[4(2-\varepsilon)/\overline{S}_d(d-1)g_{1*}\right]^{2/3} \nu_0^{2-\varepsilon}\,\overline{\mathcal{E}}^{\,\varepsilon/3} k^{\,2-d-4\varepsilon/3} R(1, g_{1*}, u)\,.$$

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Analysis of the inertial-range limit $u = m/k \to 0$ beyond RG.