Renormalization group in stochastic theory of developed turbulence 4

Low-dimensional fluctuations and improved ε expansion

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Outline

- Two-parameter expansion
- Improved ε expansion
- Two-loop results
  - Kolmogorov constant
  - Prandtl number
- Conclusion

Skewness factor in the inertial range

Large-scale pumping: $\varepsilon \to 2$, $m = \dfrac{1}{L} \to 0 \Rightarrow d_f(\mathbf{k}) \to \delta(\mathbf{k})$.

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Use independent of D_{10} quantity - the skewness factor [Adzhemyan, Antonov, Kompaniets & Vasil'ev (2003)]:

$$\mathcal{S} = S_3/S_2^{3/2}\,.$$

Unambiguous Kolmogorov constant

For $\varepsilon \geq \frac{3}{2}$ the structure function $S_2(r) \sim$ const, replace in $\mathcal{S}$ by the function with powerlike asymptotics $r\partial_r S_2(r)$ and define:

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{[-S_3(r)]^{2/3}}.$$

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Calculate Kolmogorov constant and skewness factor unambiguously as

$$C_K = \left[\frac{3Q(2)}{2}\right]\left[\frac{12}{d(d+2)}\right]^{2/3}, \quad \mathcal{S} = -\left[\frac{3Q(2)}{2}\right]^{-3/2}.$$

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Coarse-graining of finite band-width forcing always generates the local term (Forster, Nelson & Stephen, 1977).

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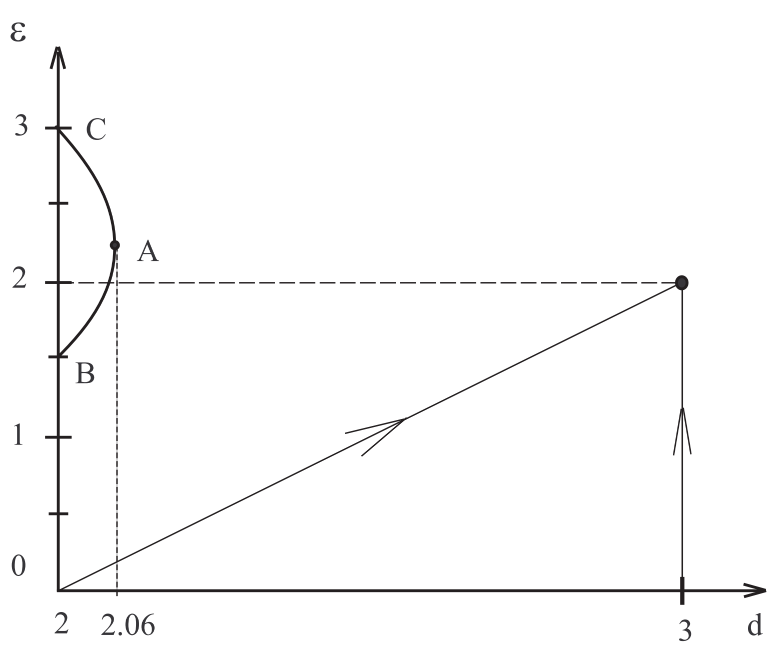

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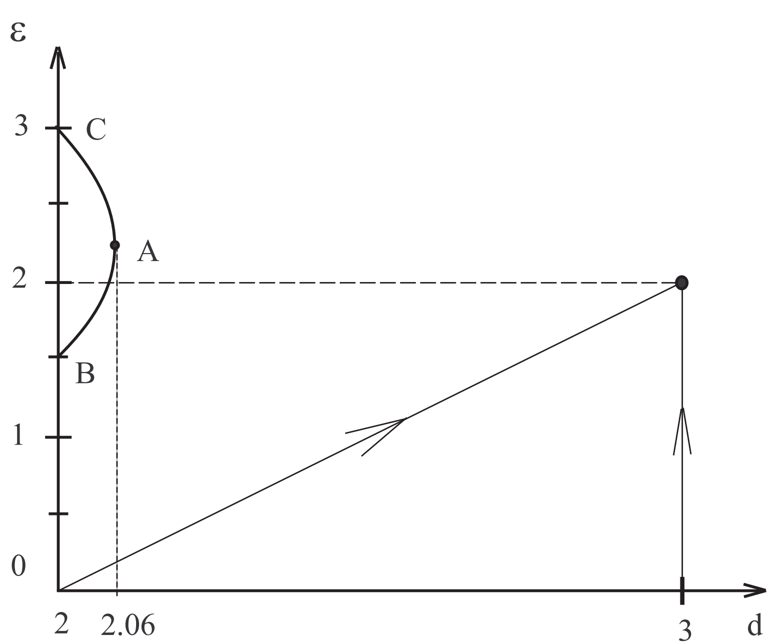

Yes, inverse energy cascade far from the linear extrapolation path.

Two-parameter expansion

Additional UV-renormalization near $d=2$ required

$$S_{\rm R} = \frac{1}{2}\mathbf{v}'\left(D_1 k^{4-d-2\varepsilon} + D_2 Z_{D_2} k^2\right)\mathbf{v}' - \mathbf{v}'\left[\partial_t \mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} - \nu Z_\nu \nabla^2 \mathbf{v}\right]$$

with $\nu_0 = \nu Z_\nu$ and

$$g_{01} = D_{10}\nu_0^{-3} = g_1 \mu^{2\varepsilon} Z_\nu^{-3}\,,\; g_{20} = D_{20}\nu_0^{-3} = g_2 \mu^{2-d} Z_{D_2} Z_\nu^{-3}\,.$$

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The RG solution [$m = 0$, UV cutoff Λ imposed]

$$G(k, g_{10}, g_{20}, \nu_0, \Lambda) = \left(D_{10}/\bar{g}_1\right)^{2/3} k^{2-d-4\varepsilon/3} R_\Lambda\left(1, \bar{g}_1, \bar{g}_2, \Lambda/k\right)\,.$$

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Near $d=2$ $\exists$ IR-stable fixed point giving rise to double expansion in ε and $2\Delta=d-2$.

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- the remainder is analytic continuation from $d < 2$.

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These are two different subsequences of the double series

$$Q(\varepsilon, d) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[2\varepsilon/(d-2)\right]^k q_{kl} \, \left[(d-2)/2\right]^l \,.$$

Improved ε expansion

Combine the information from both expansions

$$Q_{eff}^{(n)} = \varepsilon^{1/3} \left[\sum_{k=0}^{n-1} Q_k(d)\varepsilon^k + \sum_{k=0}^{n-1} \Psi_k \left(\frac{d-2}{2\varepsilon} \right) \varepsilon^k - \sum_{k,l=0}^{n-1} \left(\frac{2\varepsilon}{d-2} \right)^k q_{kl} \left(\frac{d-2}{2} \right)^l \right].$$

Subtraction term to account for double counting in the overlap region.

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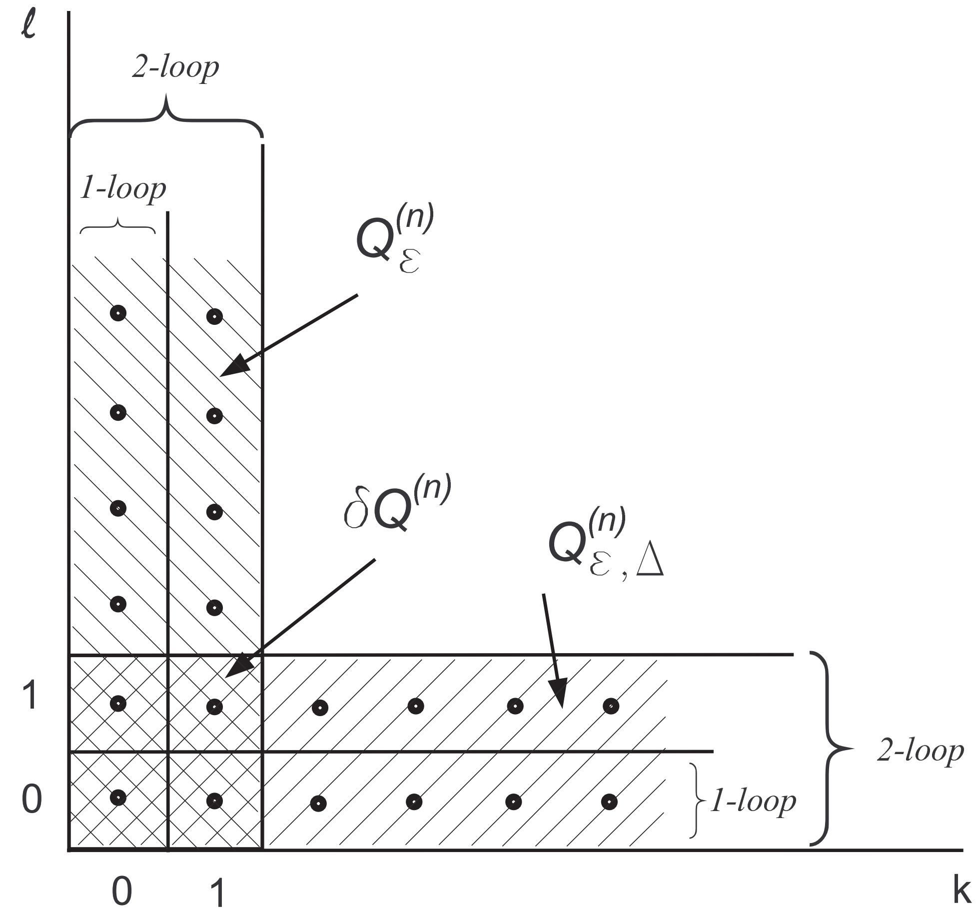

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Improved two-loop Kolmogorov constant

Comparison of one-loop and two-loop results for C_K:

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

- C_ε – ε expansion
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Recommended experimental value: $C_K = 2.0$ (Sreenivasan, 1995).

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Singular in $d-2$ contributions cancel: two-loop correction small [Adzhemyan, JH, Kim & Sladkoff (2005)]:

$$u_{eff} = u_*^{(0)}(1-0.0358\varepsilon) + O(\varepsilon^2)\,,\ u_*^{(0)} = \frac{\sqrt{43/3}-1}{2}\,,\ d=3\,.$$

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At $\varepsilon = 2$ the turbulent Prandtl number Pr_t close to accepted experimental value $\mathrm{Pr}_t \approx 0.81$:

$$\mathrm{Pr}_t^{(0)} \simeq 0.72\,, \qquad \mathrm{Pr}_t \simeq 0.77\,.$$

Ramifications of the Navier-Stokes problem

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  - LR, momentum-shell RG, weak anisotropy: Rubinstein & Barton (1987),
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  - LR, FTRG, strong anisotropy: Buša, Hnatich, JH & Horvath (1997).