Introduction to Turbulence II

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Models

- Partial Differential Equations (PDEs): Euler, Navier-Stokes, MHD, Burgers, Passive-Scalar, and BMHD equations.
- Collections of Ordinary Differential Equations (ODEs): Shell Models for fluid and MHD turbulence (e.g., GOY, SABRA, and their MHD analogues).

Euler Equation

Newton's second law of motion yields the Euler equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f}/\rho, \tag{1}$$

$\mathbf{u}$:fluid velocity; p:pressure; ρ:fluid mass density; $\mathbf{f}$:external force.

Euler

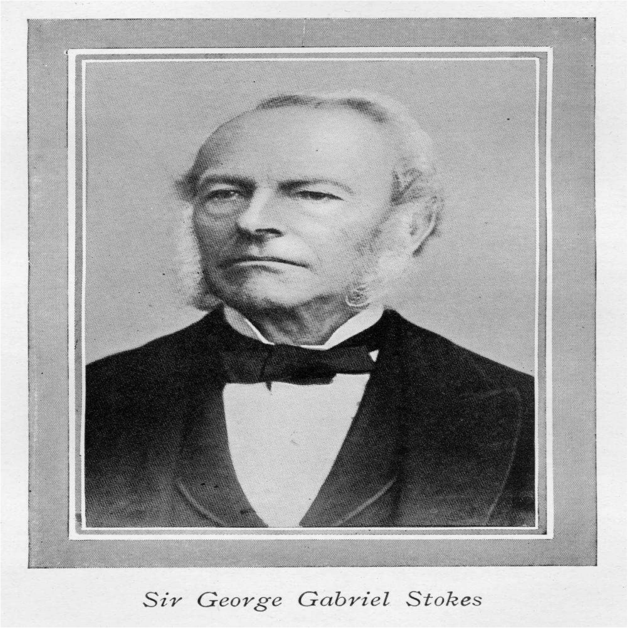
Figure 1: Leonhard Euler (1707-1783)

Incompressibility

Continuity Equation $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$

For an incompressible fluid this is replaced by the condition $\nabla \cdot \mathbf{u} = 0.$

Navier and Stokes

Claude-Louis Navier (1785-1836) and George Gabriel Stokes (1819-1903).

Navier-Stokes Equation

$$\rho \quad \left[\frac{\partial \mathbf{u}}{\partial t} + (u \cdot \nabla) u\right] = -\nabla p$$

$$+ \quad \eta \nabla^2 \mathbf{u} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f}.$$

ν: **shear viscosity;** ζ: **bulk viscosity.**

Navier-Stokes Equation

For an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

so the bulk-viscosity part does not appear in the NS equation; the pressure is determined by

$$\nabla^2 p = -\rho \nabla \cdot (\mathbf{u}.\nabla \mathbf{u}). \tag{3}$$

This is valid at low Mach numbers, i.e., if typical velocities are much less than the speed of sound in the fluid.

Incompressibility

Incompressibility leads to:

$$\begin{aligned} \partial_t \mathbf{u} &+ (\mathbf{u}\cdot\nabla)\mathbf{u} = \\ \nu &\nabla^2\mathbf{u} - \nabla p/\rho + \mathbf{f}/\rho, \end{aligned}$$

with
$\nabla\cdot\mathbf{u} = 0;$
$\nu = \eta/\rho$
the kinematic viscosity.

Shell Models

Simple ODE models for fluid or other (e.g., MHD) turbulence.
Essential ingredients of such shell models:

- Concentrate on the one-dimensional cascade of energy from low to high wavevectors.
- Label Fourier components of the velocities by a discrete set of logarithmically spaced wave vectors $k_n = k_0 \lambda^n$.

Shell Models

- The dynamical variables are the complex, scalar velocities v_n for each shell n.
- The velocity in a given shell is affected directly only by those in nearest- and next-nearest-neighbour shells.
- The system is forced at small wave vectors.
- Dissipation occurs principally at large wave vectors.
- Thus energy cascades from small to large wave vectors.

Shell Models

Advantages:

- Since they are much simpler than the Navier-Stokes equation, they can be studied in much greater detail; very large values of Re_λ and hence larger inertial ranges can be obtained.
- They yield multiscaling and multiscaling exponents akin to those seen in experiments.

Shell Models

- A shell model can be viewed as a greatly simplified, quasi-Lagrangian version of the Navier-Stokes equation since they do not have a direct sweeping effect (see below).

Shell Models

Disadvantages:

- Since the wave vectors and velocities are scalars, strictly speaking they have no vorticity or coherent structures.
- Since the velocity in a given shell is affected directly only by those in nearest- and next-nearest-neighbour shells, they do not have the analogue of the sweeping effect present in the Navier-Stokes equation.

Shell Models

- That is large eddies (i.e., v_n with small n) cannot drive, directly, small eddies (i.e., v_n with large n).
- Even though these models are much simpler than the Navier-Stokes equation, they have to be studied numerically.

GOY Shell Model

$\frac{d}{dt}v_n = iC_n - \nu k_n^2 v_n + f_n$, **with**

$$
\begin{aligned}
C_n &= \Big(a k_n v_{n+1} v_{n+2} \\
&+ b k_{n-1} v_{n-1} v_{n+1} + c k_{n-2} v_{n-1} v_{n-2}\Big)^*;
\end{aligned}
$$

GOY Shell Model

$a, b,$ and c can be fixed upto a constant by demanding that these equations satisfy all the conservation laws (analogues of energy and helicity) in the unforced, inviscid limit.

Burgers Equation

Simplest form: A one-dimensional model (not incompressible) with no pressure term.

- Galilean invariant.
- Preserves the analogue of the kinetic energy in the unforced, inviscid limit.

Burgers Equation

- Applications in cosmology, condensed-matter physics; can be used as a testing ground for ideas about turbulence.
- Can be linearised by the Hopf-Cole transformation but can still show interesting bifractal scaling.

Burgers Equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}v^2 = \nu \frac{\partial^2 v}{\partial x^2} + f.$$

Polymeric Turbulence

Navier-Stokes and FENE-P equations

Polymeric Turbulence

- Navier-Stokes(NS) with Polymer Additives:

  3D, unforced, incompressible, NS with dilute polymer solution

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \nabla \cdot \mathcal{T},$$

  where
  - $\mathbf{u}(\mathbf{x}, t)$: fluid velocity; point $\mathbf{x}$; time t;
  - ν: Kinematic viscosity of the fluid;
  - $\mathcal{T}$: polymer contribution to the fluid stress;

  $\nabla \cdot \mathbf{u} = 0$ enforces incompressibility.

Polymeric Turbulence

FENE-P Shell Model

- Kalelkar *et.al.*(2005), Benzi *et.al.*(2003):

$$\frac{du_n}{dt} = \Phi_{n,uu} - \nu_s k_n^2 u_n + \frac{\nu_p}{\tau_p(1-\sum_n |b_n|^2)}\Phi_{n,bb},$$
$$\frac{db_n}{dt} = \Phi_{n,ub} - \Phi_{n,bu} - \frac{1}{\tau_p(1-\sum_n |b_n|^2)} b_n$$

  where u_n and b_n are complex, scalar variables representing the velocity and the (normalized) polymer end-to-end vector fields, $k_n = k_0 2^n$ are the discrete wavenumbers for the shell index N

- Polymer concentration $c \equiv \nu_p/\nu_s$

Polymeric Turbulence

$$\Phi_{n,vv} = i(a_1 k_n v_{n+1} v_{n+2} + a_2 k_{n-1} v_{n+1} v_{n-1} + a_3 k_{n-2} v_{n-1} v_{n-2})]$$

$$\Phi_{n,bb} = -i(a_1 k_n b_{n+1} b_{n+2} + a_2 k_{n-1} b_{n+1} b_{n-1} + a_3 k_{n-2} b_{n-1} b_{n-2})$$

$$\Phi_{n,vb} = i(a_4 k_n v_{n+1} b_{n+2} + a_5 k_{n-1} v_{n-1} b_{n+1} + a_6 k_{n-2} v_{n-1} b_{n-2})$$

$$\Phi_{n,bv} = -i(a_4 k_n b_{n+1} v_{n+2} + a_5 k_{n-1} b_{n-1} v_{n+1} + a_6 k_{n-2} b_{n-1} v_{n-2})$$

Magnetohydrodynamics

Navier-Stokes equation and the Lorentz force.
Faraday's law and Ohm's law.
Often use incompressibility though for many physical situations this may not hold, e.g., in the solar wind and on the sun's surface.

Magnetohydrodynamics

$$
\begin{aligned}
\partial_t \vec{v} &+ (\vec{v}\cdot\nabla)\vec{v} = \nu_v \nabla^2 \vec{v} \\
-\nabla(p &+ b^2/2)/\rho + (\vec{b}\cdot\nabla)\vec{b} + \vec{f}_v/\rho, \\
\partial_t \vec{b} &= \nabla \times (\vec{v}\times\vec{b}) \\
&+ \nu_b \nabla^2 \vec{b} + \vec{f}_b,
\end{aligned}
$$

Magnetohydrodynamics

with $\vec{b}$ the magnetic field, ν_v and ν_b the fluid and magnetic kinematic viscosities and $\vec{f}_v$ and $\vec{f}_b$ the forcing terms in the velocity and magnetic-field equations.
Furthermore

$$\begin{aligned} \nabla \cdot \vec{v} &= 0; \\ \nabla \cdot \vec{b} &= 0. \end{aligned}$$

Magnetohydrodynamics

- **It is often convenient to use the Elsässer variables $\vec{z}^{\pm} \equiv \vec{v} \pm \vec{b}$.**
- **If we have a mean magnetic field $\vec{B}_0$, then, in the MHD equations, $\vec{b} \longrightarrow \vec{B}_0 + \vec{b}(\mathbf{r}, \mathbf{t})$.**
- **If $\vec{B}_0 \neq 0$ we get Alfven waves with a frequency that depends linearly on $|\vec{B}_0|$.**

Magnetohydrodynamics

Our Shell Model for 3DMHD Turbulence

$$\frac{dz_n^{\pm}}{dt} = ic_n^{\pm} - \nu_+ k_n^2 z_n^{\pm} - \nu_- k_n^2 z_n^{\pm} + f_n^{\pm},$$

with $z_n^{\pm} \equiv (v_n \pm b_n)$ **complex, scalar Elsässer variables and**

Magnetohydrodynamics

$$\begin{aligned}
c_n^{\pm} &= [a_1 k_n z_{n+1}^{\mp} z_{n+2}^{\pm} \\
&+ a_2 k_n z_{n+1}^{\pm} z_{n+2}^{\mp} \\
&+ a_3 k_{n-1} z_{n-1}^{\mp} z_{n+1}^{\pm} \\
&+ a_4 k_{n-1} z_{n-1}^{\pm} z_{n+1}^{\mp} \\
&+ a_5 k_{n-2} z_{n-1}^{\mp} z_{n-2}^{\pm} \\
&+ a_6 k_{n-2} z_{n-1}^{\mp} z_{n-2}^{\pm}]^{*}.
\end{aligned}$$

Passive Scalars

The Passive-Scalar Equation

$$
\begin{aligned}
\frac{\partial \theta}{\partial t} &+ \vec{v} \cdot \nabla \theta \\
&= \kappa \nabla^2 \theta + f(\mathbf{r}, \mathbf{t}),
\end{aligned}
$$

Passive Scalars

with $\theta(\mathbf{r},\mathbf{t})$ the passive scalar field, $\vec{v}(\mathbf{r},\mathbf{t})$ the turbulent velocity field that drives the passive scalar, and $f(\mathbf{r},\mathbf{t})$ the external force.

In some cases a stochastic velocity field is used as an input (Gawedzki, et al., RMP (2001)).

2D Navier-Stokes

- Use the stream-function ψ and vorticity ω formulation.
- The velocity $\mathbf{u}$ is a function of x and y co-ordinates only.
- The vorticity, which is a pseudo-scalar, is defined as

$$\omega \equiv \nabla \times \mathbf{u}. \tag{4}$$

2D Navier-Stokes

- The incompressibility constraint,

$$\partial_x u_x + \partial_y u_y = 0 \tag{5}$$

ensures that the velocity is uniquely determined by the stream-function, ψ, as

$$\mathbf{u} \equiv (-\partial_y \psi, \partial_x \psi). \tag{6}$$

2D Navier-Stokes

The Navier-Stokes equation :

$$\partial_t \omega - J(\psi, \omega) = \nu \nabla^2 \omega + f \quad (7)$$

$$\nabla^2 \psi = \omega, \quad (8)$$

where

$$J(\psi, \omega) \equiv (\partial_x \psi)(\partial_y \omega) - (\partial_x \omega)(\partial_y \psi). \quad (9)$$

- The source function f is the $\hat{\mathbf{z}}$ component of the curl of some force $\nabla \times \mathbf{F}$.

2D Navier-Stokes

We can also write the Navier-Stokes equation in the following equivalent form :

$$\hat{\psi} = -\frac{1}{k^2}\hat{\omega}; \tag{10}$$

$$\partial_t \omega + \partial_x (u_x \omega) + \partial_y (u_y \omega) = \nu \nabla^2 \omega + f, \tag{11}$$

where u_x and u_y are respectively the x and y component of the velocity vector $\mathbf{u}$. f is the source function or the forcing.

2D MHD

The MHD equations in 2D are given by

$$\begin{aligned} \partial_t \omega + \mathbf{v}.\nabla\omega - \mathbf{b}.\nabla j &= \nu\nabla^2\omega; \quad (12) \\ \partial_t \psi + \mathbf{v}.\nabla\psi &= \eta\nabla^2\psi. \quad (13) \end{aligned}$$

where the flux function ψ is related to the current density j via

$$j = \nabla^2\psi \quad (14)$$

Calculations

Types of Calculations

- Rigorous results on existence, etc., of solutions.
- Linear (or more sophisticated) stability analysis about simple flows.
- Renormalized perturbation theory for stochastically forced models.

Calculations

- Closures, e.g., the direct interaction approximation (DIA).
- Numerical studies, either direct numerical simulation (DNS) or various levels of turbulence modelling, e.g., $k - \epsilon$ and large-eddy smulations (LES).
- We will concentrate on DNS here.

Boundary Conditions

- The normal component of the velocity must vanish at the boundary (both for the Euler and the Navier-Stokes equations).
- For the Navier-Stokes equation the tangential component of the velocity is also controlled; for rigid boundaries we typically use the no slip condition in which the fluid, at the boundary, has a tangential velocity equal to that of the boundary.

Boundary Conditions

- The pressure also has to satisfy the boundary condition $\hat{n} \cdot \nabla p|_{\partial\Omega} = \rho[\nu \hat{n} \cdot \nabla^2 \vec{u}]|_{\partial\Omega}$.
- For studies of homogeneous, isotropic turbulence it is advantageous to use periodic boundary conditions for the pressure and all components of the velocity (and magnetic field).

Further Background

- Refs.: U. Frisch, Turbulence (Cambridge, 1996); S.B. Pope, Turbulent Flows (Cambridge, 2000); C.R. Doering and J.D. Gibbon, Applied Analysis of the Navier-Stokes Equations (Cambridge, 1995).
- Symmetries, e.g., for the Navier-Stokes equation under space and/or time translations, Galilean transformations (for infinite systems or with periodic boundary conditions), parity, rotations, and scaling.

Further Background

- Conservation laws in the unforced, inviscid limits (e.g., for the Navier-Stokes equation, conservation of momentum, energy, and helicity).
- The generalization of the above conservation laws to balance equations in the presence of forcing and dissipation.
- Eulerian and Lagrangian descriptions.

Further Background

Dimensionless Control Parameters

- Navier-Stokes: Reynolds number (if v_{rms} is held fixed) or the Grashof number (if the force is held fixed).
- MHD: Fluid and magnetic Reynolds numbers and the magnetic Prandtl number (the ratio of fluid and magnetic viscosities).

Further Background

Mathematical Issues

- Existence and smoothness of the solutions of the Navier-Stokes and Euler equations. Similar questions arise for all the equations mentioned above.
- Roughly speaking, the question is whether the solutions develop singularities, in finite time, for arbitrary (or analytic) initial data.
- For a precise statement see the web site of the Clay Mathematics Institute.

Further Background

Physicists' Perspective

- Even if there is some problem with the existence and smoothness of the solutions of the Navier-Stokes, we will not have to worry too much about it since higher-order derivatives (neglected at the Navier-Stokes level) will control it.

Multifractals

Recall K41

- As $Re \to \infty$, all possible symmetries of the NS equation, normally broken by the means of producing the turbulent flow, are restored in a statistical sense at small scales and far away from boundaries.
- In terms of velocity increments

$$\delta \mathbf{v}(\mathbf{r}+\rho, \mathbf{l}) \overset{\mathbf{law}}{=} \delta \mathbf{v}(\mathbf{r}, \mathbf{l}),$$

where equality in law denotes that the random functions on both sides of the equation have the same statistical properties (moments, PDFs, etc.) for any ρ.

Multifractals

Under the assumption on the previous slide, the turbulent flow is self similar at small scales, i.e., it possesses a unique, real, scaling exponent h

$$\delta\mathbf{v}(\mathbf{r},\lambda\mathbf{l}) \stackrel{\mathbf{law}}{=} \lambda^{\mathbf{h}}\delta\mathbf{v}(\mathbf{r},\mathbf{l}), \tag{15}$$

where λ is real, and the equality in law holds for for all $\mathbf{r}$ and all $\mathbf{l}$ and $\lambda\mathbf{l}$ smaller in magnitude than the integral scale.

Multifractals

- In particular, the order-p velocity structure function must scale as l^{ph}, thus, at the K41 level, $h = 1/3$.
- Under the assumption on the previous slide, the turbulent flow has a finite, nonvanishing mean rate of dissipation ϵ per unit mass.
- Here we must keep the integral scale l_0 and the r.m.s. velocity fluctuations v_0 fixed and let $\nu \to 0$; otherwise use $\epsilon/(v_0^3/l_0)$.

4/5 Law

Kolmogorov's 4/5 law As $Re \to \infty$, the third-order (longitudinal) structure function is

$$\langle \left(\delta v_{||}(\mathbf{r}, \mathbf{l})\right)^3 \rangle = -\frac{4}{5}\epsilon l, \tag{16}$$

for l smaller in magnitude than the integral scale.
Important, exact and nontrivial result.

Kármán-Howarth-Monin (KHM)

Define first

$$\epsilon(l) \equiv -\partial_t \frac{1}{2} \langle \mathbf{v}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}+\mathbf{l}) \rangle |_{\mathbf{NL}}, \tag{17}$$

where $\partial_t(\cdot)|_{NL}$ denotes the time rate of change from the nonlinear terms (advection and pressure) in the NS equation.

Homogeneous (but not necessarily isotropic) solutions of the NS equation satisfy

$$\begin{aligned} \epsilon(l) &= -\frac{1}{4}\nabla_{\mathbf{l}} \cdot \langle|\delta\mathbf{v}(\mathbf{l})|^{\mathbf{2}}\delta\mathbf{v}(\mathbf{l})\rangle \\ &= -\partial_t\frac{1}{2}\langle\mathbf{v}(\mathbf{r}).\mathbf{v}(\mathbf{r}+\mathbf{l})\rangle \\ &+ \langle\mathbf{v}(\mathbf{r}) \cdot \frac{\mathbf{f}(\mathbf{r}+\mathbf{l})+\mathbf{f}(\mathbf{r}-\mathbf{l})}{\mathbf{2}}\rangle \\ &+ \nu\nabla_{\mathbf{l}}^2\langle\mathbf{v}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}+\mathbf{l})\rangle, \end{aligned} \tag{18}$$

KHM

where $\nabla_{\mathbf{l}}$ denotes partial derivatives with respect to $\mathbf{l}$ and

$$\langle|\delta\mathbf{v}(\mathbf{l})|^{\mathbf{2}}\delta\mathbf{v}(\mathbf{l})\rangle \equiv \langle|\delta\mathbf{v}(\mathbf{r},\mathbf{l})|^{\mathbf{2}}\delta\mathbf{v}(\mathbf{r},\mathbf{l})\rangle.$$

If we hold $\nu > 0$ fixed and let $\mathbf{l} \to \mathbf{0}$, the LHS of KHM $\to 0$ (velocity increments vary linearly for very small l), so

KHM

$$\partial_t \frac{1}{2}\langle \mathbf{v^2} \rangle = \langle \mathbf{f}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}) \rangle + \nu \langle \mathbf{v}(\mathbf{r}) \cdot \nabla^{\mathbf{2}} \mathbf{v}(\mathbf{r}) \rangle,$$

i.e., the only change in the mean energy comes from the force input and the viscous dissipation.

Scale-by-scale energy budget equation

$$-\partial_t \mathcal{E}_K + \Pi_K = -2\nu\Omega_K + \mathcal{F}_K, \tag{19}$$

where we define the *cumulative energy, enstrophy, and energy injection*between wavenumbers 0 and K as follows:

Definitions

$$\mathcal{E}_K = \frac{1}{2} \sum_{k \le K} |\mathbf{v_k}|^2; \tag{20}$$

$$\Omega_K = \frac{1}{2} \sum_{k \le K} k^2 |\mathbf{v_k}|^2; \tag{21}$$

$$\mathcal{F}_K = \sum_{k \le K} \mathbf{f_k} \cdot \mathbf{v_k}; \tag{22}$$

Definitions

and we define the *energy flux* through the wavenumber K as follows:

$$\begin{aligned}\Pi_K &\equiv \langle \mathbf{v}^{<}_{\mathbf{K}} \cdot (\mathbf{v}^{<}_{\mathbf{K}} \cdot \nabla \mathbf{v}^{>}_{\mathbf{K}})\rangle \\ &+ \langle \mathbf{v}^{<}_{\mathbf{K}} \cdot (\mathbf{v}^{>}_{\mathbf{K}} \cdot \nabla \mathbf{v}^{>}_{\mathbf{K}})\rangle,\end{aligned}$$

Definitions

where

$$\mathbf{v}^{<}_{\mathbf{K}}(\mathbf{r}) = \int_{|\mathbf{k}|\leq\mathbf{K}} \mathbf{d^3k}\ \mathbf{e}^{\mathbf{ik\cdot r}}\ \mathbf{v_k}. \tag{23}$$

Note that

$$\Pi_K = -\partial_t \mathcal{E}_K|_{NL}. \tag{24}$$

For homogeneous turbulence:

$$\begin{aligned}\Pi_K &= -\frac{1}{8\pi^2}\int_{R^3} d^3l \frac{\sin(Kl)}{l} \\ \nabla_{\mathbf{l}} &\cdot \left[\frac{\mathbf{l}}{\mathbf{l^2}}\nabla_{\mathbf{l}}\cdot\langle|\delta\mathbf{v}(\mathbf{l})|^{\mathbf{2}}\delta\mathbf{v}(\mathbf{l})\rangle\right].\end{aligned}$$

KHM

For homogeneous and isotropic turbulence:

$$
\begin{aligned}
\Pi_K &= -\frac{1}{6\pi}\int_0^\infty dl \frac{\sin(Kl)}{l} \\
(1 &+ l\partial_l)(3 + l\partial_l) \\
(5 &+ l\partial_l)\frac{S_3(l)}{l}.
\end{aligned}
$$

KHM

For fully developed turbulence in which a statistical steady state has been established by forcing at small wavenumbers $\ll K_c$

$$\lim_{\nu \to 0} \Pi_K = \epsilon,$$

for all $K \gg K_c$, whence

KHM

$$\begin{aligned}\Pi_K &= -\int_0^\infty dx \frac{\sin x}{x} F\left(\frac{x}{K}\right) \\ &= \epsilon, \qquad \forall K \gg K_c,\end{aligned}$$

where

$$\begin{aligned}F(l) &\equiv (1 + l\partial_l)(3 + l\partial_l) \\ (5 &+ l\partial_l)\frac{S_3(l)}{6\pi l}.\end{aligned}$$

KHM

The large-K behaviour of the integral for Π_K involves the small-l behaviour of $F(l)$ and since $\int_0^\infty dx(\sin x/x) = \pi/2$

$$F(l) \approx -\frac{2}{\pi}\epsilon, \tag{25}$$

KHM

which, when substituted in the expression for $F(l)$ yields a third-order, linear ODE for $S_3(l)$ whose solution is

$$S_3(l) = -\frac{4}{5}\epsilon l, \tag{26}$$

the $4/5$ law which immediately implies

$$\begin{aligned} h &= 1/3; \\ \zeta_3 &= 1. \end{aligned}$$

Other K41 results

- Kolmogorov dissipation scale:

$$
\begin{aligned}
\eta_d &= \left(\frac{\nu^3}{\epsilon}\right)^{1/4}; \\
K_d &= \left(\frac{\nu^3}{\epsilon}\right)^{-1/4}.
\end{aligned}
\tag{27}
$$

- Energy spectrum

$$
E(k) = \epsilon^{2/3} k^{-5/3} \mathcal{C}(\eta_d k), \tag{28}
$$

with $\mathcal{C}$ a suitable scaling function.

Other K41 results

- Taylor-microscale Reynolds number $Re_\lambda \sim Re^{1/2}$, where Re is the integral-scale Reynolds number.
- The ratio of the integral and dissipation scales is $l_0/\eta_d \sim Re^{3/4}$.
- Hence for numerical simulations on a uniform grid, the minimum number of grid points per (integral scale)^3 is $N \sim Re^{9/4}$.

Intermittency

- Recall the intermittency in the plots of the energy dissipation rate per unit mass.
- Recall also the deviations of ζ_p from the K41 prediction for $p > 3$.
- Conclude, therefore, that it is plausible, though not entirely certain, that there are intermittency corrections to K41.

Intermittency

Self-similar and intermittent random functions

- Recall that K41 assumes the self-similarity of the velocity field at inertial-range scales.
- This may well be broken to yield an intermittent velocity field.

Intermittency

High-pass filtered function:

$$
\begin{aligned}
v(t) &= \int d\omega e^{i\omega t} v_\omega, \\
v_\Omega^>(t) &= \int_{\omega>\Omega} d\omega e^{i\omega t} v_\omega
\end{aligned}
\qquad (29)
$$

Intermittency

- If the flatness $F(\Omega)$ grows without bound with the filter frequency Ω, then the random function is intermittent.

$$F(\Omega) = \frac{\langle (v_\Omega^<(t))^4 \rangle}{\langle (v_\Omega^>(t))^2 \rangle^2}. \tag{30}$$

Intermittency

- Consider a random, but stationary, function and the function obtained from it by setting it to zero a fraction $(1-\gamma)$ of the total time. Then

$$\langle v_\gamma^2 \rangle = \gamma \langle v^2 \rangle, \qquad \langle v_\gamma^4 \rangle = \gamma \langle v^4 \rangle, \tag{31}$$

whence

$$F_\gamma = \frac{\langle v_\gamma^4 \rangle}{\langle v_\gamma^2 \rangle^2} = \frac{1}{\gamma} \frac{\langle v^4 \rangle}{\langle v^2 \rangle^2}. \tag{32}$$

Intermittency

- A Gaussian signal is not intermittent since $F(\Omega) = 3$, independent of Ω.
- A random function with self-similar increments and scaling exponent h is such that, for any $\lambda > 0$,

$$v^{>}_{\lambda\Omega} \stackrel{law}{=} \lambda^{-h} v^{>}_{\Omega}. \tag{33}$$

Thus, if $\Omega \to \lambda\Omega$, $F(\Omega)$ is unchanged since both numerator and denominator scale as λ^{-4h}.

Intermittency

- Intermittency can also be measured by higher-order moments like the hyperflatness (we revert to the structure functions of turbulence):

$$F_6(l) = \frac{S_6(l)}{(S_2(l))^3}, \tag{34}$$

which grows as a power as $l \to 0$ (while staying in the inertial range).

Intermittency

$$\mu \equiv 2 - \zeta_6, \tag{35}$$

which measures the deviation from ζ_6^{K41}, can be interpreted in some models of intermittency as the codimension (3 minus the dimension) of dissipative structures.

Exact results on intermittency

Assumptions

- For $Re = (l_0 v_0/\nu) \to \infty$

$$\frac{\left\langle \left(\delta v_{||}(l)\right)^{2p} \right\rangle}{v_0^{2p}} \simeq A_{2p} \left(\frac{l}{l_0}\right)^{\zeta_{2p}}, \qquad (36)$$

where A_{2p} is a positive numerical constant that is not necessarily universal.

Exact results on intermittency

- For finite Re the above scaling holds over inertial-range scales, the size of which increases with Re at least as a power law:

$$1 \gg \frac{l}{l_0} \gg (Re)^{-\alpha}, \qquad \alpha > 0. \tag{37}$$

Exact results on intermittency

Three propositions on intermittency

- For any three positive integers $p_1 \leq p_2 \leq p3$, we have the convexity inequality

$$
\begin{aligned}
(p_3 - p_1)\zeta_{2p2} &\geq \\
(p_3 - p_2)\zeta_{2p1} &+ (p_2 - p_1)\zeta_{2p3}.
\end{aligned}
$$

Exact results on intermittency

- Under the first assumption, if there exist two consecutive even numbers $2p$ and $2p+2$, such that,
-
$$\zeta_{2p} > \zeta_{2p+2}, \tag{38}$$
- then the velocity of the flow (measured in the reference frame of the mean flow) cannot be bounded.

Exact results on intermittency

- Under the second assumption and the conditions of the previous proposition, if the Mach number based on v_0 is held fixed, and Re is increased indefinitely, then the maximum Mach number of the flow also increases indefinitely.
- Note that this would violate the incompressibility condition, but, in any case, the second proposition is not consistent with a uniform (in Re) validity of the incompressible NS equation.

Intermittency Summary

- If the even-order structure functions follow power laws with exponents ζ_{2p};
- and if the incompressibility approximation does not break down at high Re;
- then the graph of ζ_{2p} versus p is concave and nondecreasing.

β *Model*

Simplest phenomenological model to incorporate a form of intermittency.

At each stage of the Richardson cascade, the number of *daughters* of a given *mother* eddy is chosen so that the fraction of volume occupied is decreased by a factor β ($0 < \beta < 1$).

β **Model**

In this model, the fraction p_l of *active* space within eddies of size $l = r^n l_0$ goes as

$$p_l = \beta^n = \beta^{\frac{ln(l/l_0)}{lnr}} = (l/l_0)^{3-D}, \tag{39}$$

where

$$3 - D \equiv \frac{ln\beta}{lnr} \tag{40}$$

D is interpreted as *fractal dimension*.

β Model

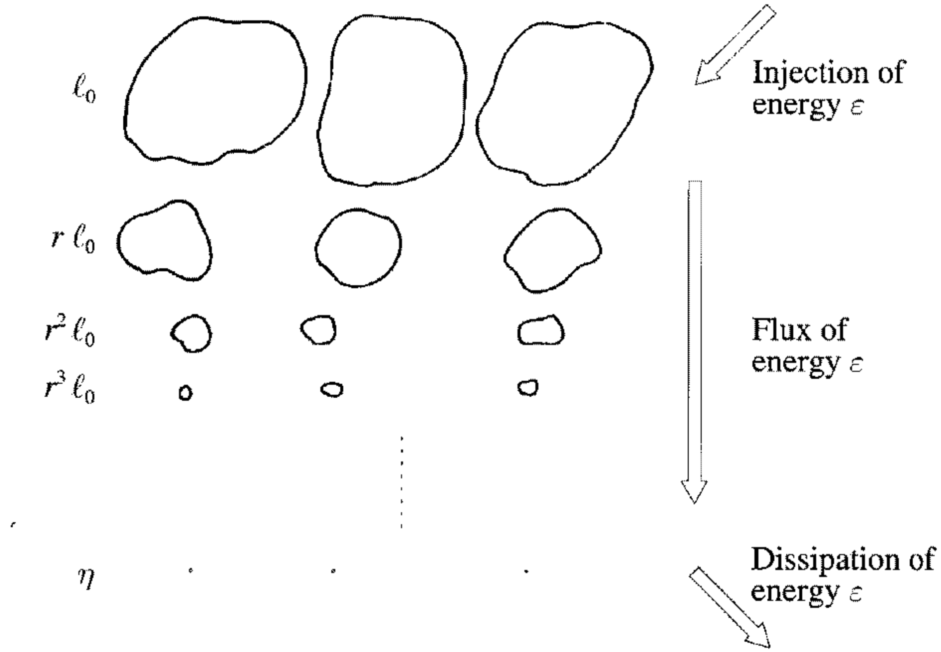


Figure 2: The cascade according to the $\beta-$model. Notice that at each step the eddies become less and less space filling.

β Model

It follows that the energy per unit mass associated with motion on scale $\sim l$ is

$$E_l \sim v_l^2 p_l = v_l^2 (l/l_0)^{3-D}. \tag{41}$$

At high Reynolds number, the inertial range energy flux is given by

$$\Pi_l' \sim \epsilon \sim \frac{v_0^3}{l_0}. \tag{42}$$

Combining we get

$$v_l \sim v_0 (\frac{l}{l_0})^{\frac{1}{3} - \frac{3-D}{3}}; \tag{43}$$

β Model

$$t_l \sim \frac{l}{v_l} \sim \frac{l_0}{v_0}\left(\frac{l}{l_0}\right)^{\frac{2}{3}+\frac{3-D}{3}}; \tag{44}$$

β Model

The velocity field has the scaling exponent

$$h = \frac{1}{3} - \frac{3-D}{3} \tag{45}$$

The order-p equal-time structure function and its scaling exponent is defined as

$$S_p(l) = \langle \delta v_l^p \rangle \sim v_0^p (\frac{l}{l_0})^{\zeta}_p, \tag{46}$$

with

$$\zeta_p = \frac{p}{3} + (3-D)(1-\frac{p}{3}), \tag{47}$$

a deviation from the K41 value.

β ***Model***

The energy spectra in the inertial range is given by

$$E(k) \propto k^{-\left(\frac{5}{3}+\frac{3-D}{3}\right)}, \tag{48}$$

which is steeper than the $k^{-5/3}$ K41 scaling.

Bifractal Model

The β-Model is equivalent to the statement that the velocity field has a scaling exponent h on a set $\mathcal{S}$ of fractal dimension D, such that h and D are related by $h = \frac{1}{3} - \frac{3-D}{3}$. A natural extension to this is the bifractal model: there are now two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ both embedded in the physical space. Near $\mathcal{S}_1$ the velocity scales as h_1, and near $\mathcal{S}_2$ the velocity scales as h_2.

Bifractal Model

Specifically, we assume the following :

$$\frac{\delta v_l(r)}{v_0} \sim (\frac{l}{l_0})^{h_1}, r \in \mathcal{S}_1, dim\mathcal{S}_1 = D_1 \tag{49}$$

$$\frac{\delta v_l(r)}{v_0} \sim (\frac{l}{l_0})^{h_2}, r \in \mathcal{S}_2, dim\mathcal{S}_2 = D_2 \tag{50}$$

Bifractal Model

The scaling exponent h_1 gives a contribution $(l/l_0)^{ph1}$ which must be multiplied by the probability $(l/l_0)^{3-D_1}$ of being within a distance l of the set $\mathcal{S}_1$ and similarly for the other exponent.So

$$S_p(l) = \mu_1(\frac{l}{l_0})^{ph1}(\frac{l}{l_0})^{3-D_1} + \mu_2(\frac{l}{l_0})^{ph2}(\frac{l}{l_0})^{3-D_2} \quad (51)$$

The power-law with the smallest exponent will dominate; we thus obtain

$$S_p(l) \propto l^{\zeta_p}, \zeta_p = min(ph_1 + 3 - D_1, ph_2 + 3 - D_2) \quad (52)$$

Multifractal Model

The velocity of a turbulent flow is assumed to possess a range of scaling exponents, $\mathcal{I} = (h_{min}, h_{max})$.
For each h in this range, there is a set $\mathcal{S}_h$ of fractal dimension $D(h)$, such that

$$\frac{\delta u_{\vec{r}}(\ell)}{u_0} \propto \left(\frac{\ell}{\ell_0}\right)^h \qquad \vec{r} \in \mathcal{S}_h$$

where u_0 is the velocity at the forcing scale ℓ_0.

Multifractal Model

$$\frac{\mathcal{S}_p(\ell)}{u_L^p} \equiv \frac{\langle \delta u^p(\ell) \rangle}{u_L^p} \propto \int_{\mathcal{I}} d\mu(h) (\frac{\ell}{L})^{[ph+3-D(h)]}$$

Multifractal Model

- The measure $d\mu(h)$ gives the weight of the different exponents.
- $(\frac{\ell}{L})^{ph}$ for power p.
- $(\frac{\ell}{L})^{3-D(h)}$ is the probability of being within a distance $\sim \ell$ of the fractal set of dimension $D(h)$.
- Steepest-descent: $\zeta_p = \inf_h [ph + 3 - D(h)]$.

Multifractal Model

Dynamic Structure Functions

$$\mathcal{F}_p(\ell, t) \propto \int_{\mathcal{I}} d\mu(h) (\frac{\ell}{L})^{\mathcal{Z}(h)} \mathcal{G}^{p,h}(\frac{t}{\tau_{p,h}}),$$

where $\mathcal{G}^{p,h}(\frac{t}{\tau_{p,h}})$ has a characteristic decay time $\tau_{p,h} \sim \ell/\delta v(\ell) \sim \ell^{1-h}$, and $\mathcal{G}^{p,h}(0) = 1$. If $\int_0^\infty t^{(M-1)} \mathcal{G}^{p,h} dt$ exists, then the order-p, degree-M, integral time scale is

Multifractal Model

$$\mathcal{T}^{I}_{p,M}(\ell) \equiv \left[\frac{1}{\mathcal{S}_p(\ell)} \int_0^\infty \mathcal{F}_p(\ell, t) t^{(M-1)} dt\right]^{(1/M)}.$$

Multifractal Model

$$\mathcal{T}^{I}_{p,1}(\ell) \equiv \left[\frac{1}{\mathcal{S}_p(\ell)}\int_0^\infty \mathcal{F}_p(\ell,t)dt\right]^{(1/M)}$$

$$\propto \left[\frac{1}{\mathcal{S}_p(\ell)}\int_{\mathcal{I}} d\mu(h)(\frac{\ell}{L})^{\mathcal{Z}(h)}\int_0^\infty dt\mathcal{G}^{p,h}(\frac{t}{\tau_{p,h}})\right]$$

$$\propto \left[\frac{1}{\mathcal{S}_p(\ell)}\int_{\mathcal{I}} d\mu(h)(\frac{\ell}{L})^{ph+3-D(h)}\ell^{1-h}\right]$$

Multifractal Model

In the last step, we have used :

$$\tau_{p,h} \sim \ell / \delta v(\ell) \sim \ell^{1-h}$$

Multifractal Model

- Corresponding Bridge Relations :

$$z^{I}_{p,1} = 1 + [\zeta_{p-1} - \zeta_p],$$

$$z^{D}_{p,2} = 1 + [\zeta_p - \zeta_{p+2}]/2.$$

- Bridge relations reduce to $z_p^{K41} = 2/3$ if we assume K41 scaling for the equal-time structure functions.

Multifractal Model

- Integral time-scale of order-p, degree-M

$$\mathcal{T}^{I}_{p,M}(\ell) \equiv \left[\frac{1}{\mathcal{S}_p(\ell)} \int_0^{\infty} \mathcal{F}_p(\ell,t) t^{(M-1)} dt\right]^{(1/M)}.$$

Multifractal Model

- From the longitudinal, time-dependent, order-p structure functions, the order-p, degree-M, integral time scale is defined as,

$$\mathcal{T}^{I}_{p,M}(r) \equiv \left[\frac{1}{\mathcal{S}_p(r)}\int_0^\infty \mathcal{F}_p(r,t)t^{(M-1)}dt\right]^{(1/M)}$$

- The integral dynamic multiscaling exponent $z^{I}_{p,M}$ is defined as

$$\mathcal{T}^{I}_{p,M}(r) \sim r^{z^{I}_{p,M}}.$$

Multifractal Model

- Similarly, the order-p, degree-M derivative time scale is defined as

$$\mathcal{T}^D_{p,M}(r) \equiv \left[\frac{1}{\mathcal{S}_p(r)} \frac{\partial^M \mathcal{F}_p(r,t)}{\partial t^M}\right]^{(-1/M)}$$

- The derivative dynamic multiscaling exponent $z^D_{p,M}$ is defined as

$$\mathcal{T}^D_{p,M}(r) \sim r^{z^D_{p,M}}.$$

Multifractal Model

- The multifractal model predicts the following bridge relations:

$$z^I_{p,M} = 1 + \frac{[\zeta_{p-M} - \zeta_p]}{M};$$

$$z^D_{p,M} = 1 + \frac{[\zeta_p - \zeta_{p+2}]}{M}.$$