Application of the quantum-field theory methods in the theory of developed turbulence

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Outline

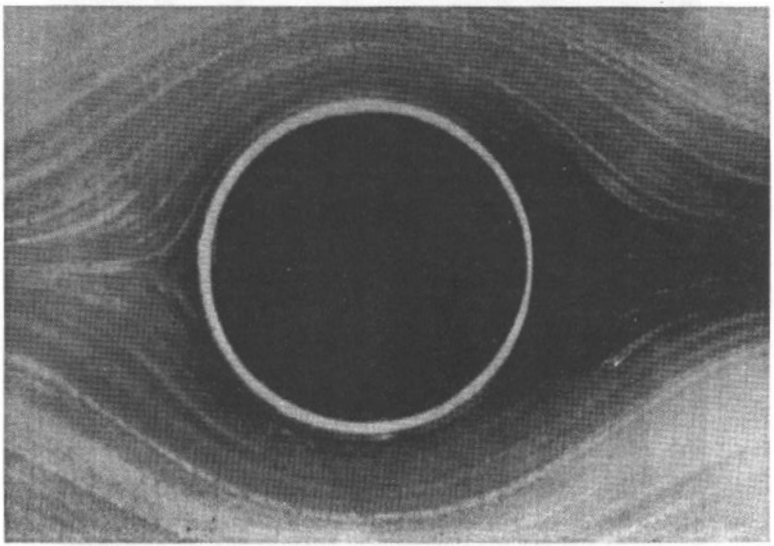
six lectures

- Short introduction to the theory of stochastic developed turbulence
- Basic terminology and technology of QFT. Schwinger equations. Divergences of graphs and ultraviolet renormalization
- Equivalence of a stochastic problem and an effective quntum field theory (field-theoretic model). Formulation of the model of stochastic developed turbulence as the field-theoretic model
- Galilean symmetry of the model, Ward identities
- Conservation laws for the energy and momentum.
- Stochastic MHD as a quantum field model

Main object for study: strongly developed turbulence

important control parameter:
Reynolds number $Re = VL/\nu_0$
typical scales are present:
L – external length (diameter of cylinder),
l – dissipation length

laminar flow: $Re = VL/\nu_0 \leq Re_{crit}$



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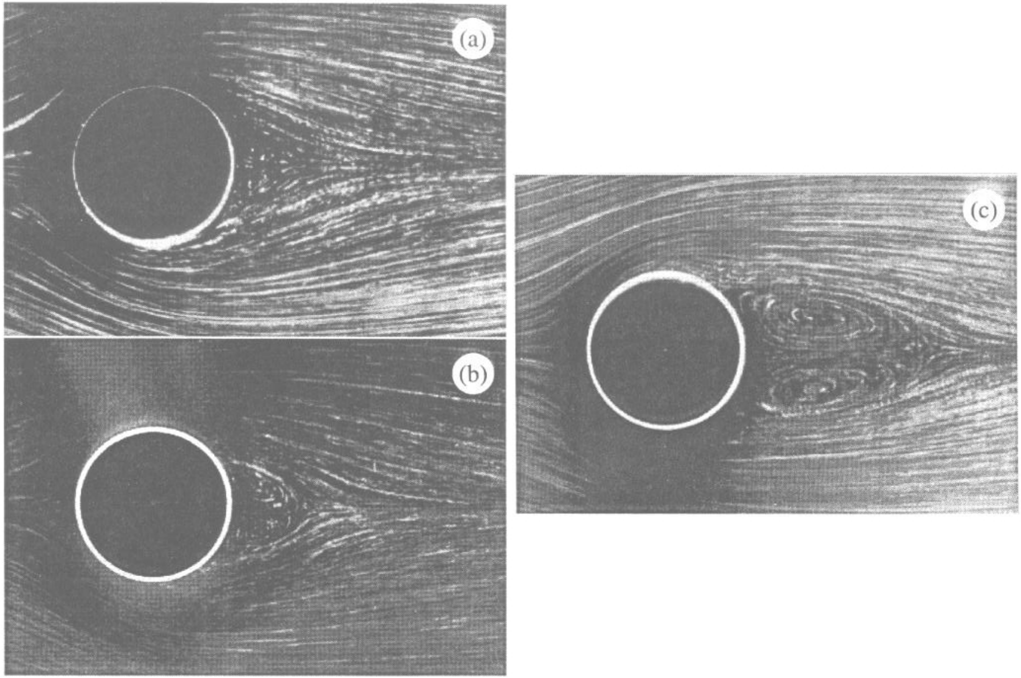
intermediate state: $Re = VL/\nu_0 \geq Re_{crit}$



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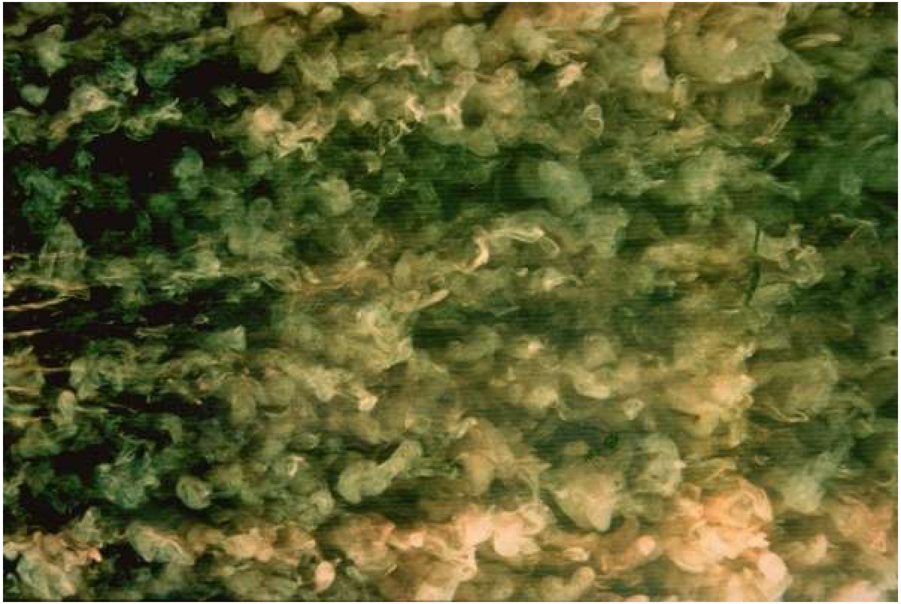
developed turbulence: $Re = VL/\nu_0 \gg Re_{crit}$



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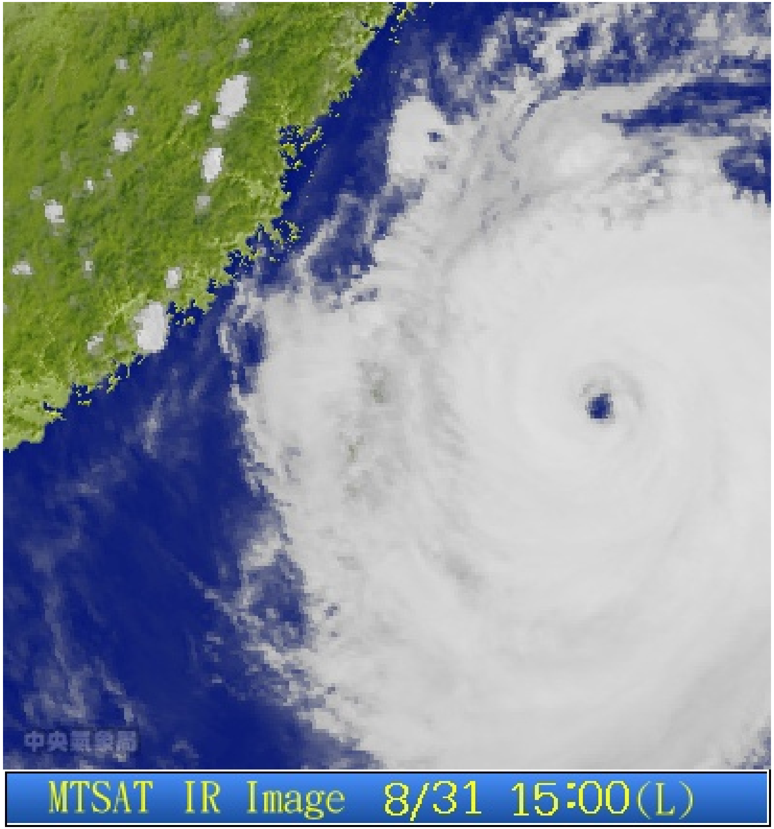
developed turbulence: $Re = VL/\nu_0 \gg Re_{crit}$



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Near threshold $Re \geq Re_{crit}$: structure of turbulent eddies first appear at L is determined by the full geometry $\Rightarrow$ remebers the details of the global structure

Problems of this type can be solved only individually for each specific system

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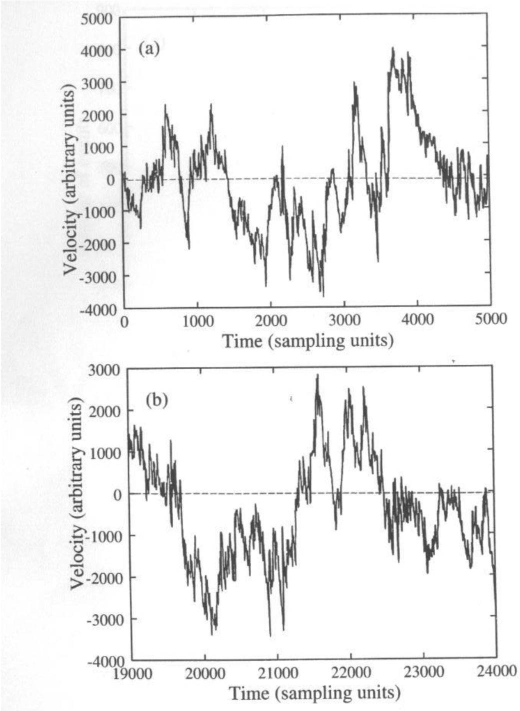
Developed turbulence: extremely irregular behaviour of velocity field in time and space:

velocity field fluctuations in time



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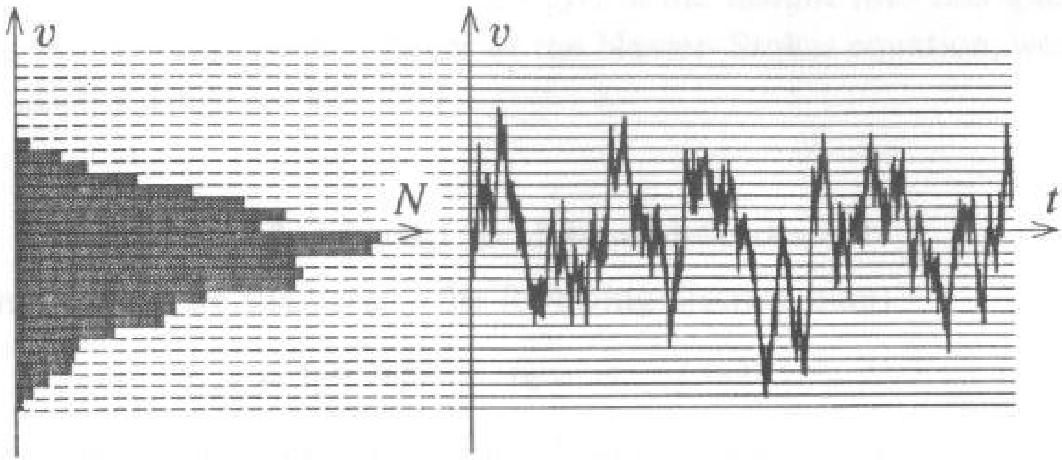
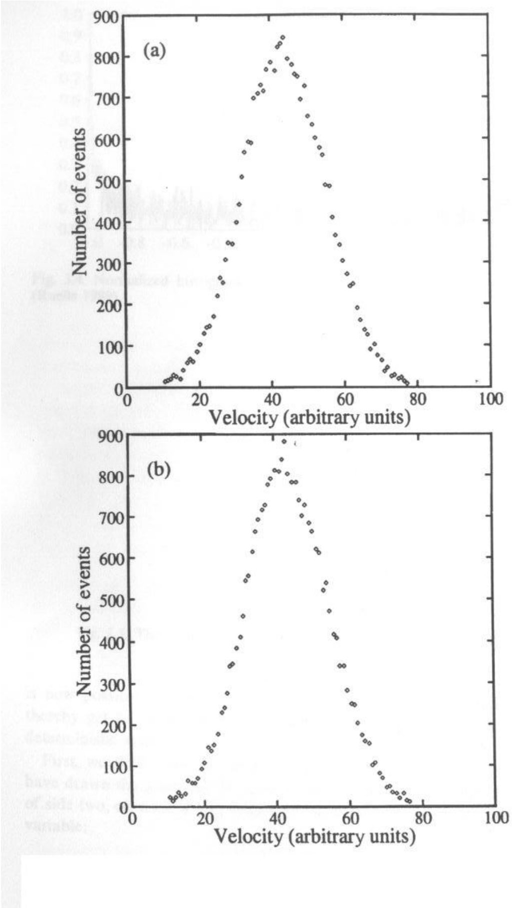
histogram for velocity fluctuations



stability of "distribution" of fluctuations in time



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- Developed turbulence: velocity field $V(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t}) + \varphi(\boldsymbol{x}, \boldsymbol{t})$ $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ smooth laminar component, $\varphi(\boldsymbol{x}, \boldsymbol{t})$ relatively small stochastic (irregular) component
- Statistic charakteristics of random field $\varphi(\boldsymbol{x}, \boldsymbol{t})$: correlation functions + various response functions = Green functions
- $Re >> Re_{crit}$, $L >> l \Rightarrow$ **inertial interval of scales** $L >> r >> l$, where we can learn Green functions and it is possible to ignore nontrivial global structure of turbulent system
- It leads to the problem **homogeneous, isotropic developed turbulence**, where largest eddies are assumed to be the energy source and the statistical correlations of random field become the main objects of interest
- Developed turbulence is observed for liquids and gases and obeys the same general laws
- Typical velocity of turbulent fluctuations: much smaller than speed of sound (Mach number is much smaller than unit) $\Rightarrow$ neglect of compressibility $\Rightarrow$ velocity field is solenoidal (transverse)

Stochastic Navier-Stokes equation for velocity fluctuations φ:

$$\partial_t \varphi + (\varphi\nabla)\varphi - \nu_0 \Delta\varphi + \nabla p = \boldsymbol{f}, \;\; \nabla_t \equiv \partial_t + (\varphi\nabla) \qquad (1)$$

pressure fluctuations p, viscosity coefficient ν_0: $(\varphi \equiv \varphi_i(\boldsymbol{x}, t), p \equiv p(\boldsymbol{x}, t))$
$\varphi\nabla \equiv \sum_i \varphi_i \nabla_i$, $i = 1 \ldots d$, scalar product

incompressible fluid with the solenoidal velocity $\nabla\varphi = 0$, unit density of fluid $\rho = 1$

$\boldsymbol{f}$ represents an external random force: mimics an interaction between average smooth velocity and fluctuations φ

Gaussian distribution with zero mean and a given pair correlator:

$$\langle f_i(x) f_j(x')\rangle \equiv D_{ij}(x, x') = \frac{\delta(t-t')}{(2\pi)^d} \int \mathrm{d}\boldsymbol{k}\, d(k) P_{ij}(\boldsymbol{k}) e^{i\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{x}')}\,, \qquad (2)$$

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d – dimension of space $\boldsymbol{x}$,

$P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j / k^2$ – the transverse projection operator in wave-vector $\boldsymbol{k}$ ($k = |\boldsymbol{k}|$) space
"Energy injection" $d(k)$:

$$d(k) = D_0 k^{4-d-2\epsilon} F(kL) \tag{3}$$

convinient to choose $D_0 = g_0 \nu_0^3$, the scaling function $F(kL)$: unit asymptotic behaviour in the range of large wave numbers (inertial interval) $kL \gg 1$
$\epsilon \geq 0$ is a free parameter of the model,
$\epsilon \geq 2$ corresponds to the energy injection into the turbulent flow from the range of the largest scales $\sim L$, or equivalently, from range of the smallest wave numbers $k \sim L^{-1}$
$d(k)$: simple relation with a physically measurable quantity – average energy dissipation rate $\overline{\mathcal{E}} = -\nu_0 \langle (\nabla_i v_j + \nabla_j v_i)^2 \rangle / 2$

$$\overline{\mathcal{E}} = \frac{d-1}{2(2\pi)^d} \int \mathrm{d}\boldsymbol{k}\, d(k)\,. \tag{4}$$

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transport phenomena in turbulent environment (advection of pollutants in the Earth's atmosphere, redistribution of heat in turbulent fluid and so on) or behaviour of the magnetic field in electrically conductive fluid $\Rightarrow$
an additional equation for the random field θ (concentration, temperature, magnetic field...)

The general form of such an equation:

$$\partial_t\theta + (\varphi\nabla)\theta - R_0\Delta\theta + H(\theta,\varphi) = \boldsymbol{f}^\theta\,, \tag{5}$$

R_0 – a "diffusion" coefficient
An external stochastic forcing $\boldsymbol{f}_\theta$: random injection of the quantity θ into the turbulent system
The form of the (non)linear term $H(\theta)$ depends on concrete models: $H = 0$ for passive scalar, $H = \theta^n$ $n = 1, 2, 3...$ for radioactive or chemically active scalar admixture, $H \equiv -(\boldsymbol{\theta}\nabla)\varphi$ for the magnetic field (N.S.: to add Lorentz force)

Main objects of interest

Green (correlation and response) functions of random fields

Physical phenomenon: Scaling in inertial interval explained in the framework of Kolmogorov phenomenological theory (K41)
For experimental study structure functions S_p are suitable:
statistical averages of equal-time powers of the projection of the velocity field φ onto the direction $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$ along two separate space coordinates $\boldsymbol{x}, \boldsymbol{x}'$

$$S_p(r) \equiv \langle[\varphi_r(\boldsymbol{x}) - \varphi_r(\boldsymbol{x}')]^p\rangle\,, \quad r \equiv |\boldsymbol{x} - \boldsymbol{x}'|\,, \quad \varphi_r \equiv \varphi\boldsymbol{r}/r \qquad (6)$$

Kolmogorov hypotheses for stationary homogeneous end isotropic developed turbulence
(only equal time correlations will be considered):

Hypothesis 1: In the region $r \ll L$ statistical distribution of the random velocity φ depends on the total pumping power (equal to the energy dissipation rate $\overline{\mathcal{E}}$), but is independent of the details of its structure, including the specific value of L

Hypothesis 2: In the region $r \gg l$ this distribution is independent of the viscosity coefficient ν_0

Particular consequence from the second hypothesis : structure functions have a simple scaling form in $r \gg l$:

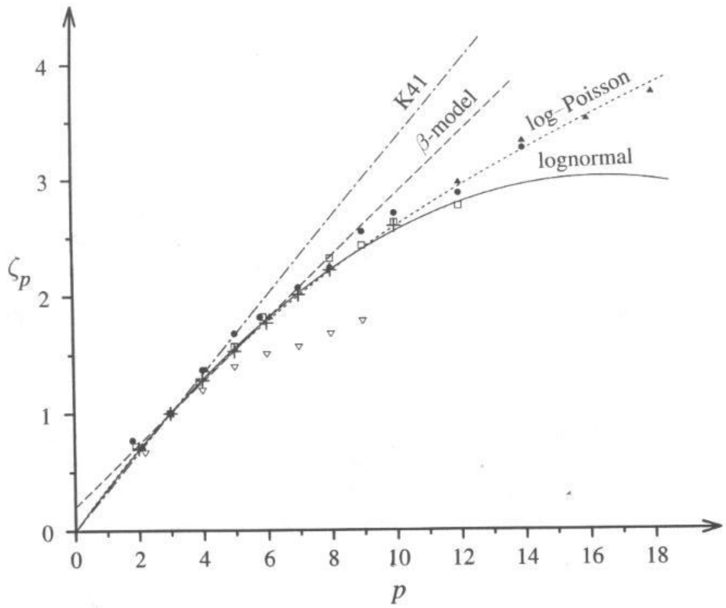
$$S_p(r) = (\overline{\mathcal{E}} r)^{p/3} f_p(r/L)\,, \qquad (7)$$

f_p are arbitrary scaling functions.
The both hypotheses:
Structure functions have a simple power form:

$$S_p(r) = C_p(\overline{\mathcal{E}} r)^{\zeta_p}\,, \qquad (8)$$

some constants C_p and the celebrated Kolmogorov exponents $\zeta_p = p/3$, which are linear functions on p

Scaling exponents ζ_p for structure functions: dependence on p



Developed turbulence and its theoretical description:

- system with infinite number of degree of freedom
- can be considered on full time axis from minus to plus infinity, in infinite space and formally for arbitrary dimension
- strongly non-linear system
- statistical system with extremely irregular behaviour of velocity field in time and space

In a certain sence it suggests the quantum field theory which is successfully applied for description of interactions of elementary particles and for critical behaviour of systems near the point of phase transition

natural ambition: to apply powerful methods of QFT to solve the principal problems of developed turbulence
to make it:

1. we have to understand how to pass over from formulation in terminology of problem based on stochastic N-S equation to the terminology of an effective quantum field model (field-theoretic model)
2. to understand what objects in QFT correspond to the statistical quatities in stochastic model (correlation functions, response functions, structure functions etc.)

Answer is:
The stochastic problems (1)- (5) can be re-formulated as quantum-field (field-theoretical functional) models with an effective action. In the framework of these models one is able to use powerful mathematical tools to derive renormalization group equations for correlation, response or structure functions of fields or more complicated quantities – the composite operators. Solutions of such equations have the scaling form in the asymptotic large-scale regions with definite exponents, which, at least, can be calculated perturbatively.

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Quantum Field Theory methods $\Leftrightarrow$ Field-Theoretic Methods

– definition of basic objects

– functional formulation, Feynman perturbation theory

– Schwinger equations

– UV renormalization

– equivalence theorem

– Galilean invariance

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Basic definitions:

we will orientate ourself on functional formulation of QFT:
quantum fields (scalar field, photon field, fermion fields for electrons, positrons, quarks etc.) $\Rightarrow$ their clasical counterpartners which appears when we carry out the quantization of arbitrary system by means of path Feynman (functional) integral
physical quantity under consideration:
a random field $\varphi(x)$ – an analogy of quantum field
Generally, $\varphi(x)$ – the set of fields with vector (discrete) indices
Features:
– we will consider euclidian space, which is natural for description of phase transitions or for classical non-relativistic systems, contrary to the pseudoeuclidian (Minkowski) space typical for relativistic systems
– for simplicity we will consider the dependence only on space variable, inclusion of time is straightforward and does not bring principal technical problems. The argument $\boldsymbol{x}$ icludes all continuous and disrete variables (indices) on which the field depends

Full correlation functions G_n of the field φ (full Green functions in field theory):

$$G_n(x_1, \ldots, x_n) = \langle \varphi(x_1) \ldots \varphi(x_n) \rangle , \; n \geq 1, \; G_0 = 1 \tag{9}$$

Averaging over a statistical ensemble (if necessary can be specified)

Spatially uniform systems: $\Rightarrow$ functions are translationally invariant $\Rightarrow$ $\langle \varphi(x) \rangle$ is independent of the spatial components of the argument x, and the higher functions depend only on their differences
The average of the product of two such fields – pair correlation function of the fluctuation field (the propagator in quantum field theory):

$$D(x, x') \equiv \langle \varphi(x)\varphi(x') \rangle - \langle \varphi(x) \rangle \langle \varphi(x') \rangle$$

Uniform system: $\Rightarrow$ depends only on the coordinate difference

Analogy in QFT: φ, operator in Gilbert space $x \equiv t, \boldsymbol{x}$, $(c = 1)$
its Green functions:

$$G_n(x_1, \ldots, x_n) = (0| \, T\{\varphi(x_1) \ldots \varphi(x_n) \, |0) \, , \; n \geq 1, \; G_0 = 1 \tag{10}$$

physical vacuum average of T–product of operators φ

The Fourier transforms of translationally invariant functions: depending only on the difference $x - x'$ of spatial coordinates in a space of arbitrary dimension d

$$F(x,x') = (2\pi)^{-d} \int dk F(k) exp[ik(x - x')]$$

$$F(k) = \int d(x - x') F(x,x') exp[ik(x' - x)]$$

coordinate and momentum representations distinguished only by the arguments

x, x' – d-dimensional spatial coordinates, k is the d-dimensional momentum (wave vector), and $k(x - x')$ is the scalar product of vectors

The fields with discrete indices $\Rightarrow$ all F are matrices in these indices

The functional formulation

Fundamental tool: functional and diagrammatic technique of quantum field theory

Green functions in field theory:

full (normalized and unnormalized),connected, 1-irreducible

Functional for full Green functions:

$$G(A) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \dots \int dx_1 \dots dx_n G_n(x_1 \dots x_n) A(x_1) \dots A(x_n) \qquad (11)$$

Argument $A(x)$ of functional G: an arbitrary function with x the same as for the field $\varphi(x)$

A functional Taylor expansion $\Rightarrow$ functions G_n are its coefficients:

$$G_n(x_1 \dots x_n) = \frac{\delta^n G(A)}{\delta A(x_1) \dots \delta A(x_n)} |_{A=0}$$

always symmetric with respect to permutations of $x_1 \dots x_n$

Connected Green functions $W_n(x_1, \ldots x_n)$

$$W(A) = \ln G(A) = \sum_n \frac{1}{n!} W_n A^n \tag{12}$$

Coefficients of the Taylor expansion of $W(A)$ analogous to (11)
$G(A) = e^{W(A)}$ expand both sides in A, and equate the coefficients of identical powers of A:

$$1 = G_0 = e^{W_0} \qquad W_0 = 0, \qquad G_1(x) = W1_(x)$$

$$G_2(x, x') = W_2(x, x') + W_1(x)W_1(x')$$

$$G_3(x, x', x'') = W_3(x, x', x'') + W_1(x)W_2(x', x'') + W_1(x')W_2(x, x'')+$$

$$+W_1(x'')W_2(x, x') + W_1(x)W1(x')W1(x'')$$

$$W_1(x) = \langle \varphi(x) \rangle \, , \, W_2(x, x') = D(x, x')$$

home excercise

One-particle irreducible functions and their generating functional

It has a physical importance, e.g. in the analysis of stability of the system above and below critical point at phase transition and its behaviour below critical point when spontaneous symmetry breaking takes place. These functions play crucial role in the analysis of UV renormalization of the theory!

Legendre transform $\Gamma(\alpha)$ of the functional W with respect to A:

$$\Gamma(\alpha) = W(A) - \alpha A,\ \alpha(x) = \frac{\delta W(A)}{\delta A(x)},\ \frac{\delta \Gamma(\alpha)}{\delta \alpha(x)} = -A(x) \tag{13}$$

$$\alpha A = \int dx \alpha(x) A(x) \tag{14}$$

Functional variables A, α are conjugate of each other, and either can be taken as the independent variable

$$-\int dx'' \frac{\delta^2 \Gamma(\alpha)}{\delta \alpha(x) \delta \alpha(x'')} \frac{\delta^2 W(A)}{\delta A(x) \delta A(x'')} = \delta(x - x') \tag{15}$$

$$\frac{\delta W(A,a)}{\delta a} = \frac{\delta \Gamma(\alpha,a)}{\delta a} \tag{16}$$

a any auxiliary numerical or functional parameter

1-irreducible Green functions

$$\Gamma_n(x_1,\ldots,x_n;\alpha) = \frac{\delta^n \Gamma(\alpha)}{\delta\alpha(x_1)\ldots\delta\alpha(x_n)} \tag{17}$$

Generating functional:

Theory with action

$$S(\varphi) = S_0(\varphi) + V(\varphi),\ S_0(\varphi) = -\frac{1}{2}\varphi K \varphi \tag{18}$$

$$Z = \int D\varphi e^{S(\varphi)}$$

expetation value of random quantity $Q(\varphi)$

$$\langle Q(\varphi) \rangle = Z^{-1} \int D\varphi Q(\varphi) e^{S(\varphi)}$$

particularly, Green functions:

$$G_n(x_1, \ldots, x_n) = Z^{-1} \int D\varphi \varphi(x_1) \ldots \varphi(x_n) e^{S(\varphi)}$$

$$G(A) = Z^{-1} \int D\varphi e^{S(\varphi) + A\varphi} \tag{19}$$

Reflections

– Mathematically: the theory of a classical random field is identical to (Euclidean) quantum field theory, for which the functional integration technique was actually developed
– Free theory with quadratic action, the Gaussian functional integrals can be calculated exactly, and the interaction V can be taken into account using perturbation theory. The convenient technique of Feynman diagrams has been developed to describe the terms of the perturbation series.
– All basic relations needed for perturbative calculations will be introduced
– All the general formulas in universal notation are valid for any field or set of fields
– The fundamental definitions involve functional (path) integrals, and so we need a precise formulation of the rules for calculating such integrals

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Technically, it is simpler to take the following formal rule for calculating Gaussian integrals as a fundamental postulate:

$$\int D\varphi e^{-\frac{1}{2}\varphi K\varphi + A\varphi} = det(K/2\pi)e^{\frac{1}{2}AK^{-1}A} \tag{20}$$

$A\varphi$ – a general linear form, $\varphi K\varphi$ – a quadratic form, linear symmetric operator K acts on the fields φ, K^{-1}– the inverse operator
All linear operators can be written as integral operators:

$$[K\varphi](x) \equiv (K\varphi)_x = \int dx' K(x,x')\varphi(x') \tag{21}$$

$$\varphi K\varphi = \int\int dx dx' \varphi(x) K(x,x')\varphi(x') \tag{22}$$

Kernel $K(x,x')$ symmetric $\Rightarrow K(x,x') = K(x',x)$

REMARK:

For translationally invariant operators (including differential operators with constant coefficients), the kernel $K(x, x')$ depends only on the difference of spatial coordinates $x - x'$. If the Fourier transform $K(k)$ is defined by a standard way, then for a differential operator it will be a simple polynomial in the momenta. A convolution of kernels then corresponds to a product of Fourier transforms without an additional coefficient, and so the inverse operator corresponds simply to $K^{-1}(k)$. All operators remain matrices in their discrete indices, if there are any.

Calculation of determinants of arbitrary operators:

$$\begin{aligned} \det(LM) &= \det L \det M,\ \det(K^\alpha) = (\det K)^\alpha, \\ \det L/detM &= \det(LM^{-1}) = \det(M^{-1}L) = \det(L/M), \quad (23) \\ \det(K^T) &= \det K, \qquad \det K = e^{tr \ln K} \quad (24) \end{aligned}$$

Let us give several useful formulas involving variational derivatives.
Variational differentiation:
basic definition

$$\delta\varphi(x)/\delta\varphi(x') = \delta(x - x')$$

VERY very usefull relations:

$$F(\delta/\delta\varphi)e^{A\varphi}\cdots = e^{A\varphi}F(A + \delta/\delta\varphi)\ldots \quad (25)$$

$$F(\delta/\delta\varphi)e^{A\varphi} = F(A)e^{A\varphi} \quad (26)$$

$$e^{A\delta/\delta\varphi}F(\varphi) = F(\varphi + A) \quad (27)$$

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Gaussian functional integral in field theory: free action plays the role of the quadratic form

Symmetric operator kernel $\Delta \equiv K^{-1}$, called the free (bare) propagator or correlator of the field, will be represented as a line in the graphs

$$S(\varphi) = S_0(\varphi) + V(\varphi),\ S_0(\varphi) = -\frac{1}{2}\varphi K\varphi,\ \Delta = \Delta^T = k^{-1} \tag{28}$$

The formal substitution $A(x) \to \delta/\delta\psi(x)$ from GI can be used to obtain the expression

$$c\int D\varphi e^{S_0(\varphi)+\varphi\frac{\delta}{\delta\psi}} = e^{\frac{1}{2}\frac{\delta}{\delta\psi}\Delta\frac{\delta}{\delta\psi}} \tag{29}$$

$$\frac{1}{2}\frac{\delta}{\delta\psi}\Delta\frac{\delta}{\delta\psi} \equiv \int\int dx dx' \frac{\delta}{\delta\psi(x)}\Delta(x,x')\frac{\delta}{\delta\psi(x')} \tag{30}$$

c – a normalization constant:

$$c^{-1} \equiv \int D\varphi e^{S_0(\varphi)} = det(K/2\pi)^{-1/2} \tag{31}$$

From (27) and (29) for an arbitrary functional $F(\psi)$:

$$e^{\frac{1}{2}\frac{\delta}{\delta\psi}\Delta\frac{\delta}{\delta\psi}}F(\psi) = c\int D\varphi F(\varphi+\psi)e^{S_0(\varphi)} \tag{32}$$

$$e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}}F(\varphi) = c\int D\varphi F(\varphi)e^{S_0(\varphi)} \tag{33}$$

N.B.:

These expressions are fundamental for calculating non-Gaussian functional integrals (32) can be used to rewrite any expression involving the exponential operator as a functional integral!!!

Generating fuctional:

$$G(A) = e^{W(A)} = c\int D\varphi e^{S(\varphi)+A\varphi} \tag{34}$$

can be rewritten in the form

$$G(A) = e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}}e^{V(\varphi)+A\varphi}|_{\varphi=0} \tag{35}$$

graphs terminology:

GF of connected Green functions
$W(A) = \ln G(A) =$ connected part of $G(A)$
contains only graphs in which every vertex is connected with remaining part of graph at least by one line

GF of one-particle inrreducible functions (defined by Legendre transform of W)
$\Gamma(\alpha) =$ IP-irreducible part of W
connected graphs:
– with amputated external lines $\wedge$ $A \to \alpha$
– all vertices in graph are connected in such a way that after breaking (eliminating) just one arbitrary internal line the graf remains connected

Composite operator F: any local construction formed from the field $\varphi(x)$ and its derivatives – $\varphi^n(x)$, $\partial_i\varphi^n(x)$, $\varphi(x)\partial^2\varphi(x)$

$F(\varphi) = F_i(x;\varphi)$– a set of composite operators
Generating functional of unnormalized full Green functions involving any number of fields φ and operators F and these functions themselves are the coefficients of the expansion of the functional in the set of sources A and a

$$G(A,a) = c\int D\varphi e^{S(\varphi)+aF(\varphi)+A\varphi}, W(A,a) = \ln G(A,a) \qquad (36)$$

$a_i(x)$ – arbitrary sources in the linear form

$$aF(\varphi) \equiv \sum_i \int dx a_i(x) F_i(x;\varphi)$$

Generating functional (first derivative of W (36) with respect to the source a at $a = 0$):

$$W_F(x;A) \equiv \langle\langle F(\varphi)\rangle\rangle \equiv \frac{\int D\varphi F(\varphi) e^{S(\varphi)+A\varphi}}{\int D\varphi e^{S(\varphi)+A\varphi}} \tag{37}$$

$$\langle F(\varphi)\rangle \equiv \frac{\int D\varphi F(\varphi) e^{S(\varphi)}}{\int D\varphi e^{S(\varphi)}} \tag{38}$$

Connected Green functions containing one operator F and arbitrary number of fields φ

$$\langle F(x)\varphi(x_1), \ldots \varphi(x_n)\rangle = \frac{\delta^n W_F(x;A)}{\delta A(x_1)\ldots\delta A(x_n)} \tag{39}$$

The Schwinger equations

Any relations expressing invariance of the measure $D\varphi$ under translations $\varphi(x) \to \varphi(x) + \omega(x)$ by arbitrary fixed functions ω belonging good-defined space with $\omega(\infty) = 0$

Such translations do not change the integration region $\Rightarrow$ the quantity

$$\int D\varphi F(\varphi + \omega)$$

is independent of ω for any F $\Rightarrow$ first variation with respect to ω gives

$$\int D\varphi \frac{\delta F(\varphi)}{\delta(x)} = 0 \tag{40}$$

$$F = e^{S(\varphi)+A\varphi}$$

$$\int D\varphi \frac{\delta}{\delta\varphi} e^{S(\varphi)+A\varphi} = 0 \tag{41}$$

$$\int D\varphi \left[\frac{\delta S(\varphi)}{\delta\varphi} + A(x)\right] e^{S(\varphi)+A\varphi} = 0 \tag{42}$$

Multiplication by φ inside the integral of GF $(G(A))$ is equivalent to differentiation of the integral with respect to A

$$\left[\frac{\delta S(\varphi)}{\delta \varphi}|_{\varphi=\delta/\delta A} + A(x)\right] G(A) = 0 \tag{43}$$

REMARK:

By substituting $G = e^W$ we can obtain the equivalent equation for $W(A)$, and from it we find the equation for $\Gamma(\alpha)$. All these equations are of finite order (for polynomial action) in the variational derivatives, and are equivalent to an infinite chain of coupled equations for the exact Green functionsthe expansion coefficients of the corresponding functionals.

home exercise: derive SE for W

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UV renormalization

QF model: specified by the action S

Green functions: infinite graph expansions

Graphs: integrals over momenta

Divergencies at large momenta $\Rightarrow$ the model contains ultraviolet (UV) divergences

Typical situation in QFT at $d = 4$ (couple constant dimensionless $\Leftrightarrow$ logarithmic theory)

Procedure of elimination of UV divergencies in graphs of Green functions: UV renormalization

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unrenormalized model: the action S with fields φ and bare parameters e_o – masses, couple constants, etc. $\Rightarrow$ generates GF with UV divergencies

renormalized model: renormalized action S_R with fields φ_R and renormalized parameters $e \Rightarrow$ generates UV finite Green fuctions

if $\varphi_R = Z_\varphi^{-1}\varphi,\ e_0 = Z_e e \quad \wedge \quad S_R(\varphi, e) = S(Z_\varphi \varphi, e_0)$ valid $\Rightarrow$

Multiplicatively renormalizable model !

Elimination of UV divergencies: it is enough to eliminate them in IP-irreducibne graphs
classification of UV divergencies by canonical dimensional counting
canonical dimension of IP-irreducible GF Γ_n : $d_{\Gamma_n} = d - n d_\varphi$
logarithmic theory $\Leftrightarrow$ dimensionless couple constant: $d_{\Gamma_n} = \delta$ – UV diverdence index
$\delta \geq 0$ corresponding graphs diverge – contain superficial divergence

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General information about the equations of stochastic dynamics (including model of stochastic developed turbulence)

Standard problem of stochastic dynamics:

$$\partial_t \varphi(x) = U(x,\varphi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x')\,, \qquad (44)$$

$\varphi(x) \equiv \varphi(\boldsymbol{x}, t)$ – a random (scalar, vector etc.) field (or set of fields)
$U(x,\varphi)$ – a given t-local functional
Random forcing $f(x)$: the Gaussian distribution with zero mean $\langle f(x) \rangle = 0$ and a given pair correlator D
Specific form of correlator dictated by the concrete physical problem under consideration

Generally: d-dimensional space, $\boldsymbol{x}$ – d-dimensional position vector

Completeness of formulation of the problem (44): convenient to add the retardation condition – reflects causality of all processes
equation for all time axis t with $\varphi \to 0$ at $t \to -\infty$ and at $|\boldsymbol{x}| \to \infty$

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for arbitrary time moment t
Quantities to be calculated are the *correlation functions* of field φ and also the *response functions* on external forcing:

$$\left\langle \frac{\delta^m \left[\varphi(x_1)...\varphi(x_n)\right]}{\delta f(x_1')...\delta f(x_n')} \right\rangle \tag{45}$$

$$\left\langle \frac{\delta\varphi(x)}{\delta f(x')} \right\rangle \tag{46}$$

Symbol $\langle \ldots \rangle$: averaging over the Gaussian distribution of the random forcing $f(x)$
– averaging over all configurations $f(x)$ with the weight $\exp\left[\frac{-fD^{-1}f}{2}\right]$

Response functions are retarded:
natural condition of causality i.e. at time t the solution φ of the equation (44) is independent of random forcing f taken at the time moment $t' > t$
Simple variant of dynamics:
a given static action $S_{st}(\varphi)$, which is a functional of a time-independent

field $\varphi(\boldsymbol{x}) \Rightarrow$ stochastic the Langevin equation:

$$\partial_t\varphi(x) = \alpha\left\{\frac{\delta S_{st}(\varphi)}{\delta\varphi(\boldsymbol{x})}|_{\varphi(\boldsymbol{x})\to\varphi(x)}\right\} + f(x), \quad \langle f(x)f(x')\rangle = 2\alpha\delta(x-x')\,, \tag{47}$$

α – the Onsager coefficient, $\delta(x-x') \equiv \delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}')$

The simplest example:
Brownian motion:

$$\partial_t r_i(t) = f_i(t), \quad \langle f_i(t)f_j(t')\rangle = 2\alpha\delta_{ij}\delta(t-t')\,, \tag{48}$$

$\varphi(x) \equiv r_i(t)$ – the coordinates of particle at time t, α –a diffusion coefficient

Important remark

The general problem (44) differs from (47) by the arbitrariness of the correlator D and the functional U, which may not be to reduce to the variational derivative of some functional

Another interesting example:

Stochastic Navier-Stokes equation

$$\partial_t \varphi(x) = \nu_0 \Delta \varphi(x) - (\varphi(x) \cdot \boldsymbol{\nabla})\, \varphi(x) - \boldsymbol{\nabla} p(x) + \boldsymbol{f}(x) \qquad (49)$$

Functional $U(x, \varphi)$:
the regular (non-random) force f_n, a linear part in field φ $L\varphi$ and a nonlinear part $n(\varphi)$

$$U(\varphi) = L\varphi + n(\varphi) + f_n \,. \qquad (50)$$

NB:

The iterative (perturbative) approach to the given problem is based on the following consideration: the linear problem is solvable exactly and the contribution of the nonlinear terms $n(\varphi)$ it is possible to include with an arbitrary precision by means of a perturbation scheme (we assume, of course, small enough weight of the nonlinearity). Due to this fact it is reasonable to rewrite equation (44) in an integral form:

$$\varphi = \Delta_{12} \left[f_n + f + n(\varphi) \right] \,, \qquad (51)$$

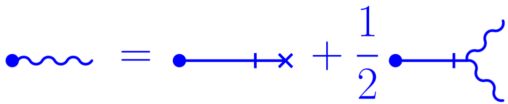
$\Delta_{12} = \Delta_{12}(x, x') \equiv (\partial_t - L)^{-1}$ – retarded Green function of linear operator $(\partial_t - L)$ ($\Delta_{12}(x, x') = 0$ for $t < t'$

Next step:
to know the specific form of nonlinear term $n(\varphi)$

Result:
Solution $\varphi(x)$ – sum of an infinite series of tree graphs (graphs without loops)
Demonstration how this method works:

$$n(x; \varphi) = v\varphi^2(x)/2 \tag{52}$$

Graphical representation $f_n = 0$:

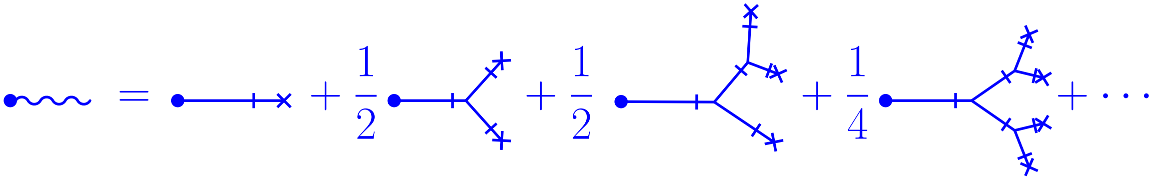


(53)

φ – wavy external line (tail), f – cross, $\Delta_{12}(x, x')$ – straight line with marked end corresponding to the argument x'
Jointing point for three graphical elements (straight lines and tails) –

vertex factor v.

Representation of $\varphi(x)$ in the form of an infinite sum of tree graphs:



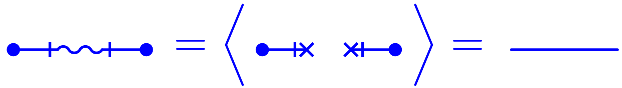
(54)

the root at the point x, crosses f on the ends of all branches.

NB:

Correlation functions are obtained by multiplying together the tree graphs in (54) for all the factors of φ and then averaging over f, which corresponds graphically to contracting pairs of crosses to form correlators D in all possible ways. This operation leads to the appearance of a new graphical element – unperturbed pair correlation function $\langle\varphi\varphi\rangle_0$, – a simple line without marks:

$$\Delta_{11} \equiv \langle\varphi\varphi\rangle_0 = \Delta_{12} D \Delta_{12}^T =$$


(55)

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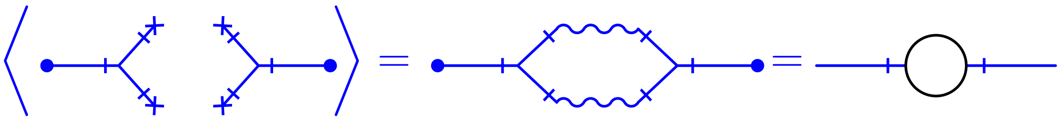
the spring – D

!!! All the crosses f should be contracted to form correlators D, and graphs with an odd number of crosses give zero when the averaging is done!!!

Correlation functions φ:– Feynman-type graphs with triple interaction vertices and two types of line: Δ_{11}, Δ_{12}.

Example:



NB:

Graphical representations of the response functions are constructed from (54) and the definition (45) in exactly the same way, and variational differentiation with respect to f corresponds graphically to removing the cross in all possible ways with the appearance of the free argument x' on the end of the corresponding line.

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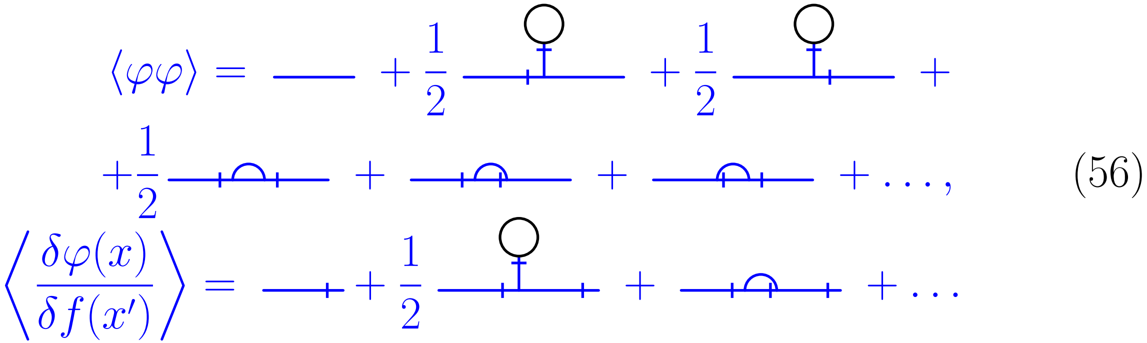
the first few graphs of the correlator $\langle\varphi\varphi\rangle$ and the response function:



In a similar manner all correlation and response functions can be obtained.

This graphical technique was first introduced and analyzed in detail for the NavierStokes equation by
H.W. Wyld, Ann. Phys. (N.Y.) 14, (1961), 143
for the general problem (44)
P.C. Martin, E.D. Siggia, H.A. Rose, Phys. Rev. A 8, (1973) 423

Reduction of the stochastic problem to a quantum field model

The graphs can be identified as Feynman graphs if we admit the existence of a second field φ' in addition to φ with zero bare correlator

$$\langle \varphi' \varphi' \rangle |_0 = 0$$

and if we interpret the line Δ_{12} as a bare correlator and the response function (46) as the exact correlator

$$\langle \varphi \varphi' \rangle .$$

The triple vertex corresponds to the interaction $V = \varphi' v \varphi \varphi / 2$. Evidently, (56) involves all the graphs of this quantum-field model without contracted lines

$$\Delta_{12} = \langle \varphi \varphi' \rangle |_0$$

and no others, so that the complete proof of the equivalence would require only showing that the symmetry coefficients coincide.

another more simple way exists how to proof the equivalence of stochastic problem with a quantum field model

Proof of equivalence of stochastic problem to a quantum-field model

Janssen, H. K. (1976) Z. Phys. B 23, 377
De Dominicis, C. and Peliti, L. (1978) Phys. Rev. B 18, 353
Adzhemyan, L. Ts., Vasilev, A. N., and Pismak, Yu. M. (1983) Teor. Math. Phys. 57, 1131

Solution of eq. (44): $\tilde{\varphi} \equiv \tilde{\varphi}(x; f)$
Generating functional $G(A)$ for correlation functions of the field $\tilde{\varphi}$

$$G(\tilde{A}) = \int Df e^{\left[-fD^{-1}f/2 + \tilde{A}\tilde{\varphi}\right]} \quad (57)$$

$\tilde{A}(x)$ – a source
Useful identity:

$$\exp(\tilde{A}\tilde{\varphi}) = \int D\varphi \delta(\varphi - \tilde{\varphi}) \exp(\tilde{A}\varphi) \quad (58)$$

Functional δ-function

$$\delta(\varphi - \tilde{\varphi}) \equiv \prod_x \delta\left[\varphi(x) - \tilde{\varphi}(x)\right] \quad (59)$$

$\tilde{\varphi}$ – unique solution of the equation (44) $\Rightarrow$

$$\varphi = \tilde{\varphi} \Leftrightarrow Q(\varphi, f) \equiv -\partial_t \varphi + U(\varphi) + f = 0 \tag{60}$$

$$\delta(\varphi - \tilde{\varphi}) = \delta\left[Q(\varphi, f)\right] \det M\,, \quad M = \frac{\delta Q}{\delta \varphi} \tag{61}$$

$M(x, x') = \delta Q(x)/\delta\varphi(x')$
Integral representation of δ-function:

$$\delta\left[Q(\varphi, f)\right] = \int D\varphi'\, e^{[\varphi' Q(\varphi, f)]} \tag{62}$$

φ' - auxiliary field

$$G(\tilde{A}) = \int\int D\varphi D\varphi' \det M \exp\left[\varphi' D\varphi'/2 + \varphi'\left(-\partial_t\varphi + U(\varphi)\right) + \tilde{A}\varphi\right] \tag{63}$$

Contribution of the determinant $\det M \equiv \det\left[\frac{\delta Q}{\delta\varphi}\right]$:

$$M = M_0 + M_1,\ M_0 = -\partial_t + L = -\Delta_{12}^{-1},\ M_1 \equiv \frac{\delta n(\varphi)}{\delta\varphi} \tag{64}$$

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$$\det M = \det M_0 \det [1 - \Delta_{12} M_1]$$

An infinite series

$$\ln \det [1 - \Delta_{12} M_1] = -\mathrm{tr}\, [\Delta_{12} M_1 + \Delta_{12} M_1 \Delta_{12} M_1 + \cdots] \qquad (65)$$

A diagrammatic representation: closed loops

$$\begin{aligned} \ln \det(1 - M) &= \mathrm{tr}\, \ln(1 - M) = \\ &= -\mathrm{tr}\left[M + M^2/2 + M^3/3 + \cdots\right] \end{aligned} \qquad (66)$$

Retardation condition $\Rightarrow$ step function $\theta(t)$ $\Rightarrow$ all closed multi-loops vanish

The first one-loop term remains

$$\int\int dx dx' \, \Delta_{12}(x, x') M_1(x', x) \qquad (67)$$

t-locality of the functional U $\Rightarrow$ kernel $M_1(x', x) = \delta(t' - t)\tilde{M}_1(x', x)$

Closed single-line loop contains the equal-time function (propagator) $\Delta_{12}(x, x')$, ($t = t'$) - not defined

$$(\partial_t - L)\Delta_{12}(x, x') = \delta(x - x') \qquad (68)$$

$$\Delta_{12}(x, x') = \theta(t - t')R(x, x') \tag{69}$$

$$R(x, x')|_{t=t'} = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{70}$$

$$(\partial_t - L)R(x, x') = 0 \tag{71}$$

$$\Delta_{12}(x, x')|_{t=t'+0} = \delta(\boldsymbol{x} - \boldsymbol{x}'), \qquad \Delta_{12}(x, x')|_{t=t'-0} = 0 \tag{72}$$

Various definitions of the propagator $\Delta_{12}(x, x')|_{t=t'}$
Usually $\Delta_{12}(x, x')|_{t=t'} = 1/2$
Reasonable to put $\Delta_{12}(x, x')|_{t=t'} = 0$
$\det M$ – an irrelevant constant

Statement:

An arbitrary stochastic problem (44) is equivalent to the field theoretic model with double number of fields $\phi \equiv \varphi, \varphi'$ and with the action functional

$$S(\phi) = \int \int dx dx' \varphi'(x) D(x,x') \varphi'(x')/2 + + \int dx \varphi'(x) \left[-\partial_t \varphi(x) + U(\varphi(x)) \right] \tag{73}$$

Language of the field-theoretic model:
Green functions – functional "averages" over the fields ϕ with "weight" $\exp S$
Generating functional:

$$G(A) = \int D\phi \, e^{[S(\phi)+A\phi]}$$

$$A\phi \equiv \int dx \left[\tilde{A}(x)\varphi(x) + A'(x)\varphi'(x) \right], \; D\phi \equiv D\varphi D\varphi'$$

Response function $\langle \varphi\varphi' \rangle$ –

$$\langle \varphi(x)\varphi'(x') \rangle = \frac{\delta^2 G(A)}{\delta \tilde{A}(x)\delta A'(x')}|_{A=0} = \int D\phi\, \varphi(x)\varphi'(x')e^{S(\phi)}$$

Source A' – regular (non-random) force added to the equation
These full Green functions (also connected and IP-irreducible) are the same as their partners defined by (45) in the framework of the original stochastic problem!!!

Action

$$\begin{aligned} S(\phi) &= \int\int dx dx'\, \varphi'(x)D(x,x')\varphi'(x')/2 + \\ &+ \int dx\, \varphi'(x)\left[-\partial_t\varphi(x) + L\varphi(x) + n\left(\varphi(x)\right)\right] \end{aligned} \tag{74}$$

$S_0(\varphi,\varphi')$ – quadratic in fields part, $V(\varphi,\varphi') = \varphi' n(\varphi)$ – interaction part

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Symmetric form of S_0

$$S_0(\varphi,\varphi') \; = \; -\frac{1}{2}\varphi K\varphi \equiv$$
$$\equiv -\frac{1}{2}\begin{pmatrix}\varphi \\ \varphi'\end{pmatrix}\begin{pmatrix}0 & (\partial_t - L)^T \\ \partial_t - L & -D\end{pmatrix}\begin{pmatrix}\varphi \\ \varphi'\end{pmatrix}, \quad (75)$$

$K^T(x,x') = K(x',x)$
Inverse matrix $\Delta = K^{-1}$ – a set of propagators

$$\Delta_{12} = \Delta_{21}^T = (\partial_t - L)^{-1}, \;\; \Delta_{11} = \Delta_{12} D \Delta_{21}, \;\; \Delta_{22} = 0,$$

$$\Delta_{ik}(x,x') = \langle \varphi_i(x), \varphi_k(x')\rangle_0 \;, \quad (76)$$

$\varphi_1 \equiv \varphi$, $\varphi_2 \equiv \varphi'$
Propagator Δ_{12} – retarded $\Rightarrow \Delta_{21} = \Delta_{12}^T$ – advanced
Symmetric propagator $\Delta_{11} = \Delta_{11}^T$ contains both (retarded and advanced) contributions. Interaction part: vertices with one field φ' and two or more fields φ (dictated by the concrete form of the nonlinear terms)

Standard Feynman graphs of Green functions with lines (propagators) defined by S_0 and vertices defined by $V(\varphi)$ by means of the Wick theorem:

$$G(A) = e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}} e^{V(\varphi)+A\varphi}|_{\varphi=0}, \tag{77}$$

Δ – the matrix of propagators

$$\delta/\delta\varphi_i \Delta_{ik} \delta/\delta\varphi_k \equiv \int dx dx' \delta/\delta\varphi_i(x) \Delta_{ik}(x,x') \delta/\delta\varphi_k(x')$$

–a universal differential operation

Expansion of both exponents in (77) $\Rightarrow$ all Green functions as infinite series of Feynman graphs

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Let us demonstrate how these general rules work in the theory of developed turbulence.
According to the MSR mechanism the stochastic model described by Eq. (1)-(3) is equivalent to the field-theoretic model with action (74), which in this case is of the following form:

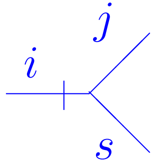
$$
\begin{aligned}
S(\varphi, \varphi') \; = \; & \int\int dx dx' \, \varphi'_i(x) D_{ij}(x, x') \varphi'_j(x') + \\
& + \int dx \, \varphi'(x) \cdot [-\partial_t \varphi(x) + \nu_0 \Delta \varphi(x) - (\varphi(x) \cdot \nabla) \varphi(x)]
\end{aligned}
$$

the auxiliary vector field φ' is solenoidal $(\nabla \varphi' = 0)$ like the velocity field φ, Feynman rules for calculation of the Green functions:
the propagators (lines) Δ and vertices V:

 $= < \varphi_i \varphi_j >_0 \equiv \Delta_{ij}^{\varphi\varphi}$

 $= < \varphi_i \varphi'_j >_0 \equiv \Delta_{ij}^{\varphi\varphi'}$

 $= < \varphi'_i \varphi'_j > \equiv \Delta_{ij}^{\varphi'\varphi'} = 0 \,,$



$$\equiv V_{ijs}.$$

The explicit form of propagators can be obtained from the quadratic part of action (78) and in the frequency-wave-vector and time-wave-vector representation have the form:

$$\begin{aligned}
\Delta_{ij}^{\varphi\varphi}(\boldsymbol{k},\omega_k) &= \frac{P_{ij}D(k)}{(i\omega_k+\nu_0k^2)(-i\omega_k+\nu_0k^2)}\,,\\
\Delta_{ij}^{\varphi\varphi}(\boldsymbol{k},t'-t) &= \frac{P_{ij}D(k)}{2\nu_0k^2}e^{-\nu_0k^2|t'-t|}\,,\\
\Delta_{ij}^{\varphi\varphi'}(\boldsymbol{k},\omega_k) &= \frac{P_{ij}}{(-i\omega_k+\nu_0k^2)}\,,\\
\Delta_{ij}^{\varphi\varphi'}(\boldsymbol{k},t'-t) &= \theta\,(t'-t)\,e^{-\nu_0k^2(t'-t)}P_{ij} \qquad (78)
\end{aligned}$$

$P_{ij}=\delta_{ij}-k_ik_j/k^2$ is transverse projector due to incompressibility
The propagator $\Delta^{\varphi\varphi}$ represents the leading approximation of the pair correlation function of the velocity field $W_{2ij}=\langle\varphi_i\,\varphi_j\rangle$, which in the wave-vector representation is proportional to the kinetic energy spec-

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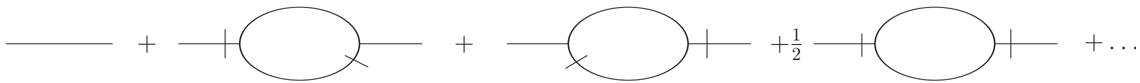
trum $E(k)$

The vertex V is given by the non-linear part of (78) and in the frequency-wave-vector representation has the form:

$$V_{ijs}^{k} = i(k_j\delta_{is} + k_s\delta_{ij}), \tag{79}$$

$\boldsymbol{k}$ denotes the wave-vector transferred by field φ_i'
Diagrammatic representation of the pair correlation function $W_2 = \langle\varphi\,\varphi\rangle$ in the one loop-approximation (first order in coupling constant g_0):

Pair correlation function of velocity field with one-loop precision



Ward identities

The Ward identities are various relations following from an exact or approximate symmetry of the action. The simplest is the statement that the Green functions are invariant, and it is conveniently stated in the language of the corresponding generating functionals. In the theory of stochastic turbulence Ward identities allow us to derive the relations between Green functions and composite operators.

Consider the Galilean transformations of fields $\phi \equiv \varphi', \varphi$, $\phi \to \phi_v$:

$$\varphi_v(x) = \varphi(x_v) - v(t),\ \varphi'_v(x) = \varphi'(x_v)$$

$$x \equiv \boldsymbol{x}, t;\ x_v \equiv \boldsymbol{x} + u(t), t;\ u(t) = \int_{-\infty}^{t} dt' v(t') = \int_{-\infty}^{\infty} dt' \theta(t-t') v(t'),$$

where $v(t)$ is parameter of transformation. It is arbitrary velocity (vector function) depending only on time and falls well-enough as $|t| \to \infty$.

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transformation for action:

$$S(\phi_v) = S(\phi) + \varphi' \partial_t v$$

Galilean invariance for turbulence reads:

$$G(A) = G(A_v) \Rightarrow \delta_v G(A) = 0$$

functional integral measure is invariant: $D\phi = D\phi_v$

$$\int D\phi \delta_v e^{S(\phi_v) + A\phi_v + aF_v} = 0$$

$$\int D\phi \left[\phi' \partial_t v + A\delta_v \phi + a\delta_v F\right] e^{S(\phi) + A\phi + aF} = 0 \qquad (80)$$

$$\delta_v \varphi(x) = u\partial\varphi(x) - v,\ \delta_v \varphi'(x) = u\partial\varphi'(x),$$

$$\delta_v \partial_t \varphi(x) = u\partial\partial_t \varphi(x) + v\partial\varphi(x) - \partial_t v,\ \partial \equiv \nabla \equiv \frac{\partial}{\partial \boldsymbol{x}}$$

$a = 0 \Rightarrow$ Ward identities for Green fuctions

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terms linear in $a \Rightarrow$ Ward identities for composite operators

Ward identities for Green functions

$$\langle\langle \phi' \partial_t v + A\partial\phi \rangle\rangle = 0$$

(80) $\Rightarrow$

$$\int dt \int d\boldsymbol{x} v(t) \left\langle\left\langle -\partial_t \phi'(x) + \int_{-\infty}^{t} A(\boldsymbol{x}, t') \frac{\partial \phi(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - A_{\varphi}(x) \right\rangle\right\rangle = 0$$

v is arbitrary $\Rightarrow$

$$\int d\boldsymbol{x} \left\langle\left\langle -\partial_t \phi'(x) + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x}, t') \frac{\partial \phi(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - A_{\varphi}(x) \right\rangle\right\rangle = 0$$

$$\phi \quad \text{in} \quad \langle\langle \quad \rangle\rangle \Leftrightarrow \frac{\delta}{\delta A}$$

$$\int d\boldsymbol{x} \left\langle\left\langle -\partial_t \frac{\delta}{\delta A_{\varphi'}(x)} + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x}, t') \frac{\partial}{\partial \boldsymbol{x}} \frac{\delta}{\delta A(\boldsymbol{x}, t')} - A_{\varphi}(x) \right\rangle\right\rangle = 0$$

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$$G = e^W$$

$$\int d\boldsymbol{x} \left[-\partial_t \frac{\delta W}{\delta A_{\varphi'}(x)} + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x}, t') \frac{\partial}{\partial \boldsymbol{x}} \frac{\delta W}{\delta A(\boldsymbol{x}, t')} - A_\varphi(x) \right] = 0$$

in term of generating functional of one-particle irreducible functions:

$$\Gamma(\alpha) = W(A) - \alpha A, \quad \alpha(x) = \frac{\delta W(A)}{\delta A(x)}, \quad A(x) = -\frac{\delta \Gamma(\alpha)}{\delta \alpha(x)}$$

$$\int d\boldsymbol{x} \left[-\partial_t \alpha_{\varphi'}(x) + \int_{-\infty}^{\infty} \theta(t-t') \frac{\delta \Gamma(\alpha)}{\delta \alpha(\boldsymbol{x}, t')} \frac{\partial \alpha(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - \frac{\delta \Gamma(\alpha)}{\delta \alpha_\varphi(x)} \right] = 0 \tag{81}$$

$$\Gamma(\alpha) = \alpha_{\varphi'} \Gamma_{\varphi'\varphi} \alpha_\varphi + \frac{1}{2} \alpha_{\varphi'} \Gamma_{\varphi'\varphi\varphi} \alpha_\varphi^2 + \dots$$

$$\alpha_{\varphi'} \Gamma_{\varphi'\varphi\varphi} \alpha_\varphi^2 \equiv \int\int\int dx_1 dx_2 dx_3 \alpha_{\varphi'_i}(x_1) \Gamma_{\varphi'_i \varphi_s \varphi_l}(x_1, x_2, x_3) \alpha_{\varphi_s}(x_2) \alpha_{\varphi_l}(x_3)$$

$$\Gamma_{\varphi'_i \varphi_s}(x_1, x_2) \equiv \Gamma_{is}(x_1, x_2), \ \Gamma_{\varphi'_i \varphi_s \varphi_l}(x_1, x_2, x_3) \equiv \Gamma_{isl}(x_1, x_2, x_3)$$

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Substitution this expansion to the (81) gives

$$\int d\boldsymbol{x}\ \Gamma_{isl}(x_1,x_2,x)+\left[\theta(t-t_1)\frac{\partial}{\partial x_{1l}}+\theta(t-t_2)\frac{\partial}{\partial x_{2s}}\right]\Gamma_{is}(x_1,x_2)=0 \tag{82}$$

integration over t: important!!! due to translation invariance of $\Gamma_{is}(x_1,x_2)$ $\frac{\partial}{\partial \boldsymbol{x}_2}=-\frac{\partial}{\partial \boldsymbol{x}_1}$ and from it we obtain in (82)

$$\theta(t-t_1)-\theta(t-t_2)\Rightarrow$$

after integration (82) over t we obtain final expression in time-coordinate reprezentation

$$\int dx\ \Gamma_{isl}(x_1,x_2,x)+(t_2-t_1)\frac{\partial\Gamma_{is}(x_1,x_2)}{\partial x_{1l}}=0$$

Ward identity in wave number - frequency representation $p\equiv \boldsymbol{k},\omega$:

$$\Gamma_{is}(x_1,x_2)=\frac{1}{(2\pi)^{2+d}}\int dp\ \Gamma_{is}(p)e^{ip(x_1-x_2)}$$

$$\Gamma_{isl}(x_1,x_2,x_3)=$$

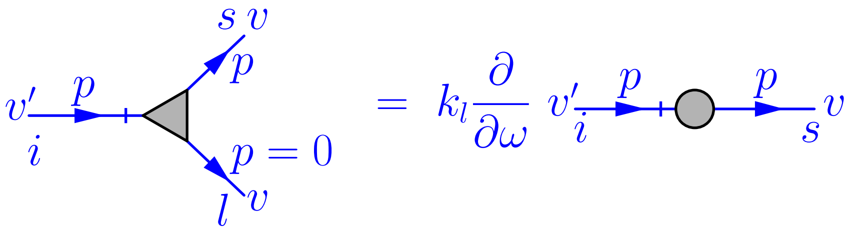
$$\frac{1}{(2\pi)^{3(2+d)}}\int\int\int dp_1 dp_2 dp_3 \delta(p_1+p_2+p_3)\Gamma_{isl}(p_1,p_2,p_3)e^{i(p_1x_1+p_2x_2+p_3x_3)}$$

$$\int dx\ \Gamma_{isl}(x_1,x_2,x) = \frac{1}{(2\pi)^{(2+d)}}\int dp\ \Gamma_{isl}(p,p,0)e^{ip(x_1-x_2)}$$

Finally

$$\Gamma_{isl}(p,p,0) = k_l \frac{\partial}{\partial\omega}\Gamma_{is}(p)$$

Graphic representation:



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Conservation laws of energy – momentum in stochastic hydrodynamics

Conservation laws of energy – momentum: important role to understand the processes of redistribution and transfer of energy end momenta in turbulent flow

Stochastic hydrodynamics: energy, momentum, their flows – random quantities constructed on velocity and its derivatives

In field theory: composite operators

Swinger equations $\Rightarrow$ Conservation laws

$$\int D\phi \frac{\delta}{\delta\phi} e^{S(\phi)+A\phi} = 0, \quad \phi \equiv \varphi', \varphi \tag{83}$$

$\Rightarrow$ Composite operator (random quantity) inside of brackets equals to zero

$$\frac{\delta S(\phi)}{\delta\phi} + A(x) = 0 \tag{84}$$

First Swinger equation in field model of stochastic hydrodynamics:

$$\int D\phi \frac{\delta}{\delta \varphi'_i(x)} e^{S(\phi)+A\phi} = 0 \tag{85}$$

$$A_i^{\varphi'} + D_{is}\varphi'_s - \partial_t \varphi_i + \nu_0 \Delta \varphi_i + (\varphi\partial)\varphi_i - \partial_i p = 0 \tag{86}$$

non-local composite operator (pressure):

$$p = -\frac{\partial_l \partial_s}{\Delta}(\varphi_l \varphi_s) \tag{87}$$

Second Swinger equation in field model of stochastic hydrodynamics:

$$\int D\phi \frac{\delta}{\delta \varphi'_i(x)} \varphi_i(x) e^{S(\phi)+A\phi} = 0 \tag{88}$$

$$\varphi A^{\varphi'} + \varphi D\varphi' - \varphi\partial_t \varphi + \varphi\nu_0 \Delta\varphi + \varphi(\varphi\partial)\varphi - (\varphi\partial)p = 0 \tag{89}$$

These relations – equations of composite operators: conservation laws of energy and momentum

$$\partial_t \varphi_i + \partial_k \Pi_{ik} = D_{ik}\varphi'_k + A_i^{\varphi'} \tag{90}$$

$$\partial_t E + \partial_i S_i = \varepsilon + \varphi D \varphi' + \varphi A^{\varphi'} \tag{91}$$

conservation laws for densities of conserved quatities (per unit mass, $\rho = 0$):
φ_i – momentum density, $E = \varphi^2/2$ – energy density

tensor of momentum flow density:

$$\Pi_{ik} = p\delta_{ik} + \varphi_i\varphi_k - \nu_0(\partial_i\varphi_k + \partial_k\varphi_i) \tag{92}$$

vector of energy flow density:

$$S_i = p\varphi_i - \nu_0\varphi_k(\partial_i\varphi_k + \partial_k\varphi_i) + \frac{1}{2}\varphi_i\varphi^2 \tag{93}$$

energy dissipation rate: tensor of momentum flow density:

$$\varepsilon = \frac{1}{2}\nu_0(\partial_i\varphi_k + \partial_k\varphi_i)^2 \tag{94}$$

Averaging equations over ϕ with weight $expS(\phi)$ $\Rightarrow$ energy momentum balance equations

energy balance equation at vanishing external non-random forcing $A^{\varphi'}$:

$$\partial_t \langle E \rangle + \partial_i \langle S_i \rangle = -\langle \varepsilon \rangle + \langle \varphi D \varphi' \rangle \tag{95}$$

homogeneous an isotropic theory at vanishing external forcing $\Rightarrow$ mean value $\langle F(x) \rangle$ of arbitrary composite operator $F(x)$ independent of x i.e. constant $\Rightarrow$ all its derivatives vanish:

$$0 = -\bar{\varepsilon} + \int\int \boldsymbol{x}' dt' \langle \varphi_i(x)\varphi'_s(x') \rangle D_{is}(x, x'),\ \bar{\varepsilon} = \langle \varepsilon \rangle \tag{96}$$

$$D_{ij}(x, x') = \frac{\delta(t - t')}{(2\pi)^d} \int \mathrm{d}\boldsymbol{k}\, D(k) P_{ij}(\boldsymbol{k}) e^{i\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}')} \equiv \delta(t - t') d_{is}(\boldsymbol{x}, \boldsymbol{x}') \tag{97}$$

$$\bar{\varepsilon} = \int \boldsymbol{x}' \langle \varphi_i(x)\varphi'_s(x') \rangle |_{t=t'} d_{is}(\boldsymbol{x}, \boldsymbol{x}') \tag{98}$$

$$\langle \varphi_i(x)\varphi'_s(x') \rangle |_{t=t'} = \frac{1}{2}\delta(\boldsymbol{x} - \boldsymbol{x}') P_{is} \tag{99}$$

Stationary homogeneous isotropic developed turbulence

pair correlator of random force f is expressed via the energy dissipation rate ($=$ pumping power):

$$\bar{\varepsilon} = \frac{1}{2} tr\, d(\boldsymbol{x}, \boldsymbol{x}) \qquad (100)$$

or

$$\bar{\varepsilon} = \frac{d-1}{2(2\pi)^d} \int \mathrm{d}\boldsymbol{k}\, d(k), \quad P_{ii}(\boldsymbol{k}) = tr P(\boldsymbol{k}) = d-1, \quad k \equiv \boldsymbol{k} \qquad (101)$$