Outline

- Equivalence of a stochastic problem and an effective quntum field theory

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UV renormalization

QF model: specified by the action S

Green functions: infinite graph expansions

Graphs: integrals over momenta

Divergencies at large momenta $\Rightarrow$ the model contains ultraviolet (UV) divergences

Typical situation in QFT at $d = 4$ (couple constant dimensionless $\Leftrightarrow$ logarithmic theory)

Procedure of elimination of UV divergencies in graphs of Green functions: UV renormalization

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unrenormalized model: the action S with fields φ and bare parameters e_o – masses, couple constants, etc. $\Rightarrow$ generates GF with UV divergencies

renormalized model: renormalized action S_R with fields φ_R and renormalized parameters e $\Rightarrow$ generates UV finite Green fuctions

if $\varphi_R = Z_\varphi^{-1}\varphi$, $e_0 = Z_e e \quad \wedge \quad S_R(\varphi, e) = S(Z_\varphi \varphi, e_0)$ valid $\Rightarrow$

Multiplicatively renormalizable model !

Elimination of UV divergencies: it is enough to eliminate them in IP-irreducibne graphs
classification of UV divergencies by canonical dimensional counting
canonical dimension of IP-irreducible GF Γ_n : $d_{\Gamma_n} = d - n d_\varphi$
logarithmic theory $\Leftrightarrow$ dimensionless couple constant: $d_{\Gamma_n} = \delta$ – UV diverdence index
$\delta \geq 0$ corresponding graphs diverge – contain superficial divergence

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General information about the equations of stochastic dynamics (including model of stochastic developed turbulence)

Standard problem of stochastic dynamics:

$$\partial_t \varphi(x) = U(x,\varphi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x')\,, \tag{1}$$

$\varphi(x) \equiv \varphi(\boldsymbol{x},t)$ – a random (scalar, vector etc.) field (or set of fields)
$U(x,\varphi)$ – a given t-local functional
Random forcing $f(x)$: the Gaussian distribution with zero mean $\langle f(x) \rangle = 0$ and a given pair correlator D
Specific form of correlator dictated by the concrete physical problem under consideration

Generally: d-dimensional space, $\boldsymbol{x}$ – d-dimensional position vector

Completeness of formulation of the problem (1): convenient to add the retardation condition – reflects causality of all processes
equation for all time axis t with $\varphi \to 0$ at $t \to -\infty$ and at $|\boldsymbol{x}| \to \infty$

for arbitrary time moment t

Quantities to be calculated are the *correlation functions* of field φ and also the *response functions* on external forcing:

$$\left\langle \frac{\delta^m \left[\varphi(x_1)...\varphi(x_n)\right]}{\delta f(x'_1)...\delta f(x'_n)} \right\rangle \tag{2}$$

$$\left\langle \frac{\delta\varphi(x)}{\delta f(x')} \right\rangle \tag{3}$$

Symbol $\langle \ldots \rangle$: averaging over the Gaussian distribution of the random forcing $f(x)$

– averaging over all configurations $f(x)$ with the weight $\exp\left[\frac{-fD^{-1}f}{2}\right]$

Response functions are retarded:

natural condition of causality i.e. at time t the solution φ of the equation (1) is independent of random forcing f taken at the time moment $t' > t$

Simple variant of dynamics:

a given static action $S_{st}(\varphi)$, which is a functional of a time-independent field $\varphi(\boldsymbol{x}) \Rightarrow$ stochastic the Langevin equation:

$$\partial_t\varphi(x) = \alpha\left\{\frac{\delta S_{st}(\varphi)}{\delta\varphi(\boldsymbol{x})}|_{\varphi(\boldsymbol{x})\to\varphi(x)}\right\} + f(x), \quad \langle f(x)f(x')\rangle = 2\alpha\delta(x-x'), \tag{4}$$

α – the Onsager coefficient, $\delta(x-x') \equiv \delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}')$

The simplest example:
Brownian motion:

$$\partial_t r_i(t) = f_i(t), \quad \langle f_i(t)f_j(t')\rangle = 2\alpha\delta_{ij}\delta(t-t'), \tag{5}$$

$\varphi(x) \equiv r_i(t)$ – the coordinates of particle at time t, α –a diffusion coefficient

Important remark

The general problem (1) differs from (4) by the arbitrariness of the correlator D and the functional U, which may not be to reduce to the variational derivative of some functional

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Another interesting example:

Stochastic Navier-Stokes equation

$$\partial_t \varphi(x) = \nu_0 \Delta \varphi(x) - (\varphi(x) \cdot \boldsymbol{\nabla})\, \varphi(x) - \boldsymbol{\nabla} p(x) + \boldsymbol{f}(x) \qquad (6)$$

Functional $U(x, \varphi)$:
the regular (non-random) force f_n, a linear part in field φ $L\varphi$ and a nonlinear part $n(\varphi)$

$$U(\varphi) = L\varphi + n(\varphi) + f_n \,. \qquad (7)$$

NB:

The iterative (perturbative) approach to the given problem is based on the following consideration: the linear problem is solvable exactly and the contribution of the nonlinear terms $n(\varphi)$ it is possible to include with an arbitrary precision by means of a perturbation scheme (we assume, of course, small enough weight of the nonlinearity). Due to this fact it is reasonable to rewrite equation (1) in an integral form:

$$\varphi = \Delta_{12} \left[f_n + f + n(\varphi) \right] \,, \qquad (8)$$

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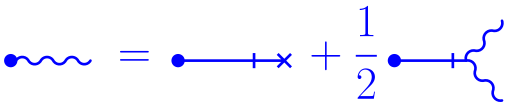
$\Delta_{12} = \Delta_{12}(x, x') \equiv (\partial_t - L)^{-1}$ – retarded Green function of linear operator $(\partial_t - L)$ ($\Delta_{12}(x, x') = 0$ for $t < t'$

Next step:
to know the specific form of nonlinear term $n(\varphi)$

Result:
Solution $\varphi(x)$ – sum of an infinite series of tree graphs (graphs without loops)
Demonstration how this method works:

$$n(x; \varphi) = v\varphi^2(x)/2 \tag{9}$$

Graphical representation $f_n = 0$:



(10)

φ – wavy external line (tail), f – cross, $\Delta_{12}(x, x')$ – straight line with marked end corresponding to the argument x'
Jointing point for three graphical elements (straight lines and tails) –

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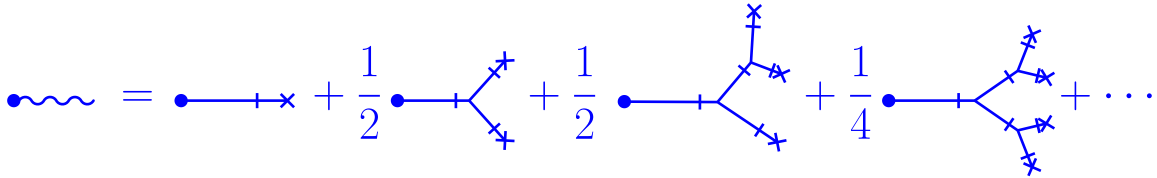
vertex factor v.

Representation of $\varphi(x)$ in the form of an infinite sum of tree graphs:



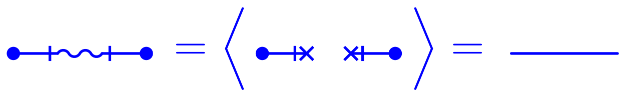
(11)

the root at the point x, crosses f on the ends of all branches.

NB:

Correlation functions are obtained by multiplying together the tree graphs in (11) for all the factors of φ and then averaging over f, which corresponds graphically to contracting pairs of crosses to form correlators D in all possible ways. This operation leads to the appearance of a new graphical element – unperturbed pair correlation function $\langle\varphi\varphi\rangle_0$, – a simple line without marks:

$$\Delta_{11} \equiv \langle\varphi\varphi\rangle_0 = \Delta_{12} D \Delta_{12}^T =$$



(12)

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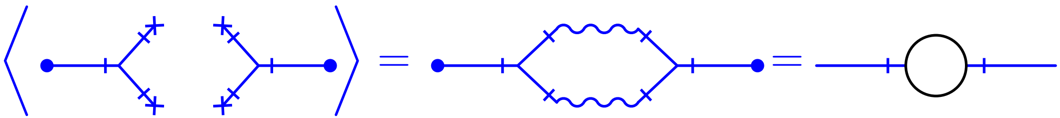
the spring – D

!!! All the crosses f should be contracted to form correlators D, and graphs with an odd number of crosses give zero when the averaging is done!!!

Correlation functions φ:– Feynman-type graphs with triple interaction vertices and two types of line: Δ_{11}, Δ_{12}.

Example:



NB:

Graphical representations of the response functions are constructed from (11) and the definition (2) in exactly the same way, and variational differentiation with respect to f corresponds graphically to removing the cross in all possible ways with the appearance of the free argument x' on the end of the corresponding line.

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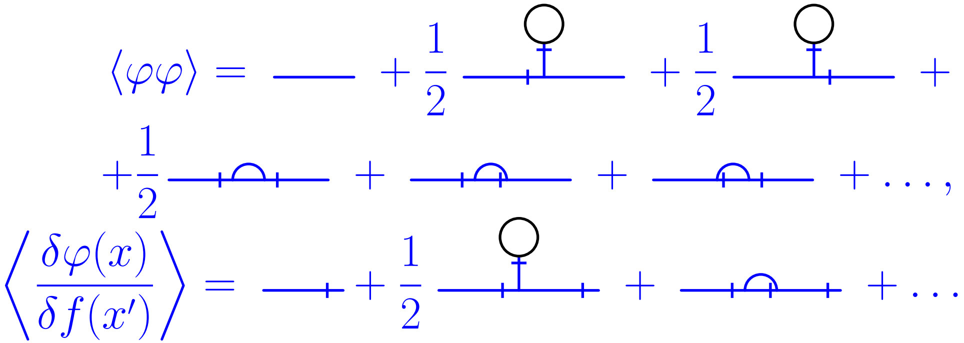
the first few graphs of the correlator $\langle\varphi\varphi\rangle$ and the response function:



(13)

In a similar manner all correlation and response functions can be obtained.

This graphical technique was first introduced and analyzed in detail for the NavierStokes equation by
H.W. Wyld, Ann. Phys. (N.Y.) 14, (1961), 143
for the general problem (1)
P.C. Martin, E.D. Siggia, H.A. Rose, Phys. Rev. A 8, (1973) 423

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Reduction of the stochastic problem to a quantum field model

The graphs can be identified as Feynman graphs if we admit the existence of a second field φ' in addition to φ with zero bare correlator

$$\langle \varphi' \varphi' \rangle \mid_0 = 0$$

and if we interpret the line Δ_{12} as a bare correlator and the response function (3) as the exact correlator

$$\langle \varphi \varphi' \rangle \, .$$

The triple vertex corresponds to the interaction $V = \varphi' v \varphi \varphi / 2.$ Evidently, (13) involves all the graphs of this quantum-field model without contracted lines

$$\Delta_{12} = \langle \varphi \varphi' \rangle \mid_0$$

and no others, so that the complete proof of the equivalence would require only showing that the symmetry coefficients coincide.

another more simple way exists how to proof the equivalence of stochastic problem with a quantum field model

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Janssen, H. K. (1976) Z. Phys. B 23, 377
De Dominicis, C. and Peliti, L. (1978) Phys. Rev. B 18, 353
Adzhemyan, L. Ts., Vasilev, A. N., and Pismak, Yu. M. (1983) Teor. Math. Phys. 57, 1131
solution of the Navier Stokes equation (??)

Solution of eq. (1): $\tilde{\varphi} \equiv \tilde{\varphi}(x; f)$
Generating functional $G(A)$ for correlation functions of the field φ

$$G(\tilde{A}) = \frac{\int Df \exp\left[-fD^{-1}f/2 + \tilde{A}\tilde{\varphi}\right]}{\int Df \exp\left[-fD^{-1}f/2\right]}, \tag{14}$$

$\tilde{A}(x)$ – a source Useful identity:

$$\exp(\tilde{A}\tilde{\varphi}) = \int D\varphi\delta(\varphi - \tilde{\varphi}) \exp(\tilde{A}\varphi) \tag{15}$$

Functional δ-function

$$\delta(\varphi - \tilde{\varphi}) \equiv \prod_x \delta\left[\varphi(x) - \tilde{\varphi}(x)\right] \tag{16}$$

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$\tilde{\varphi}$ – unique solution of the equation (1) $\Rightarrow$

$$\varphi = \tilde{\varphi} \Leftrightarrow Q(\varphi, f) \equiv -\partial_t \varphi + U(\varphi) + f = 0 \tag{17}$$

$$\delta(\varphi - \tilde{\varphi}) = \delta\left[Q(\varphi, f)\right] \det M \,, \quad M = \frac{\delta Q}{\delta \varphi} \tag{18}$$

$M(x, x') = \delta Q(x)/\delta\varphi(x')$
Integral representation of δ-function:

$$\delta\left[Q(\varphi, f)\right] = \int D\varphi' \, \exp\left[\varphi' Q(\varphi, f)\right]. \tag{19}$$

φ' - auxiliary field

$$G(\tilde{A}) = \int\int D\varphi D\varphi' \det M \exp\left[\varphi' D \varphi'/2 + \varphi'\left(-\partial_t \varphi + U(\varphi)\right) + \tilde{A}\varphi\right]. \tag{20}$$

Contribution of the determinant $\det M \equiv \det\left[\frac{\delta Q}{\delta\varphi}\right]$:

$$M = M_0 + M_1, \; M_0 = -\partial_t + L = -\Delta_{12}^{-1}, \; M_1 \equiv \frac{\delta n(\varphi)}{\delta \varphi}\,, \tag{21}$$

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$$\det M = \det M_0 \det\left[1 - \Delta_{12} M_1\right] .$$

An infinite series

$$\ln \det\left[1 - \Delta_{12} M_1\right] = -\mathrm{tr}\left[\Delta_{12} M_1 + \Delta_{12} M_1 \Delta_{12} M_1 + \cdots\right] . \qquad (22)$$

A diagrammatic representation: closed loops

$$\begin{aligned} \ln\, \det(1 - M) &= \mathrm{tr}\, \ln(1 - M) = \\ &= -\mathrm{tr}\left[M + M^2/2 + M^3/3 + \cdots\right] , \end{aligned} \qquad (23)$$

Retardation condition $\Rightarrow$ step function $\theta(t)$ $\Rightarrow$ all closed multi-loops vanish

The first one-loop term remains

$$\int\int dx dx' \, \Delta_{12}(x, x') M_1(x', x) \qquad (24)$$

t-locality of the functional U $\Rightarrow$ kernel $M_1(x', x) = \delta(t' - t)\tilde{M}_1(x', x)$

Closed single-line loop contains the equal-time function (propagator) $\Delta_{12}(x, x')$, ($t = t'$) - not defined

$$(\partial_t - L)\Delta_{12}(x, x') = \delta(x - x') \qquad (25)$$

$$\Delta_{12}(x,x') = \theta(t-t')R(x,x')\,, \tag{26}$$

$$R(x,x')|_{t=t'} = \delta(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{27}$$

$$(\partial_t - L)R(x,x') = 0\,. \tag{28}$$

$$\Delta_{12}(x,x')|_{t=t'+0} = \delta(\boldsymbol{x}-\boldsymbol{x}'), \qquad \Delta_{12}(x,x')|_{t=t'-0} = 0 \tag{29}$$

Various definitions of the propagator $\Delta_{12}(x,x')|_{t=t'}$
Usually $\Delta_{12}(x,x')|_{t=t'} = 1/2$
Reasonable to put $\Delta_{12}(x,x')|_{t=t'} = 0$
$\det M$ – an irrelevant constant

Statement:

An arbitrary stochastic problem (1) is equivalent to the field theoretic model with double number of fields $\varphi \equiv \varphi, \varphi'$ and with the action

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functional

$$S(\varphi) = \int\int dx dx' \varphi'(x) D(x,x')\varphi'(x')/2 + \int dx \varphi'(x)\left[-\partial_t\varphi(x) + U\left(\varphi(x)\right)\right]$$

Language of the field-theoretic model:
Green functions – functional "averages" over the fields φ with weight $\exp S$ (S
Generating functional:

$$G(A) = \int D\varphi \exp\left[S(\varphi) + A\varphi\right], \;\; A\varphi \equiv \int dx \left[\tilde{A}(x)\varphi(x) + A'(x)\varphi'(x)\right] \tag{30}$$

Response function $\langle\varphi\varphi'\rangle$ –

$$\langle\varphi(x)\varphi'(x')\rangle = \frac{\delta^2 G(A)}{\delta\tilde{A}(x)\delta A'(x')}|_{A=0} = \int D\varphi\, \varphi(x)\varphi'(x') e^{S(\varphi)}$$

$D\varphi \equiv D\varphi D\varphi'$. Source A' – regular (non-random) force added to the equation

These full Green functions (also connected and IP-irreducible) are the same as their partners defined by (2) in the framework of the original stochastic problem!!!

Action

$$S(\varphi) = \int\int dx dx' \, \varphi'(x) D(x,x') \varphi'(x')/2 + \\ + \int dx \, \varphi'(x) \left[-\partial_t \varphi(x) + L\varphi(x) + n\left(\varphi(x)\right) \right] \qquad (31)$$

$S_0(\varphi, \varphi')$ – quadratic in fields part, $S_{int} = \varphi' n(\varphi)$ – interaction part

Symmetric form of S_0

$$S_0(\varphi, \varphi') = -\frac{1}{2}\varphi K \varphi \equiv \\ \equiv -\frac{1}{2} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \begin{pmatrix} 0 & (\partial_t - L)^T \\ \partial_t - L & -D \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}, \qquad (32)$$

$K^T(x, x') = K(x', x)$

Inverse matrix $\Delta = K^{-1}$ – a set of propagators

$$\Delta_{12} = \Delta_{21}^T = (\partial_t - L)^{-1}, \;\; \Delta_{11} = \Delta_{12} D \Delta_{21}, \;\; \Delta_{22} = 0,$$

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$$\Delta_{ik}(x,x') = \langle \varphi_i(x), \varphi_k(x') \rangle_0 \ , \tag{33}$$

$\varphi_1 \equiv \varphi$, $\varphi_2 \equiv \varphi'$
Propagator Δ_{12} – retarded $\Rightarrow$ $\Delta_{21} = \Delta_{12}^T$ – advanced
Symmetric propagator $\Delta_{11} = \Delta_{11}^T$ contains both (retarded and advanced) contributions. Interaction part: vertices with one field φ' and two or more fields φ (dictated by the concrete form of the nonlinear terms)

Standard Feynman graphs of Green functions with lines (propagators) defined by S_0 and vertices defined by $V(\varphi)$ by means of the Wick theorem:

$$G(A) = e^{\frac{1}{2}\frac{\delta}{\delta\varphi}\Delta\frac{\delta}{\delta\varphi}} e^{V(\varphi)+A\varphi}|_{\varphi=0}, \tag{34}$$

Δ – the matrix of propagators

$$\delta/\delta\varphi_i \Delta_{ik} \delta/\delta\varphi_k \equiv \int dx dx' \delta/\delta\varphi_i(x) \Delta_{ik}(x,x') \delta/\delta\varphi_k(x')$$

–a universal differential operation
Expanding both exponents in (34) $\Rightarrow$ all Green functions as infinite se-

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ries of Feynman graphs.

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Let us demonstrate how these general rules work in the theory of developed turbulence. According to the aforementioned MSR mechanism the stochastic model described by Eq. (??)-(??) is equivalent to the field-theoretic model with action (31), which in this case is of the following form:

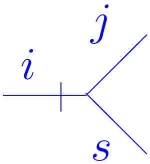
$$S(\varphi,\varphi') = \int\int dx dx' \, \frac{g_0\nu_0^3}{2}\varphi_i'(x)\tilde{D}_{ij}(x,x')\varphi_j'(x') + \\ + \int dx \, \varphi'(x)\cdot[-\partial_t\varphi(x) + \nu_0\Delta\varphi(x) - (\varphi(x)\cdot\boldsymbol{\nabla})\varphi(x)]$$

where the auxiliary vector field φ' is solenoidal $(\nabla\varphi' = 0)$ like the velocity field φ, and ν_0 is bare (molecular) viscosity coefficient. To distinguish it from the renormalized (turbulent) viscosity ν, which appears in the process of the renormalization procedure (see below) we mark it and other analogous parameters (e.g. the coupling constant g_0 in (??)) by the subscript "zero". The noise D_{ij} (see Eq. (??)) we rewrote in form $g_0\nu_0^3\tilde{D}_{ij}$ which is more convenient for further analysis. In case of an incompressible fluid the contribution of pressure into the action (35) vanishes due to the condition $(\nabla\varphi' = 0)$. By means of the general operation (34) one obtains Feynman rules for the propagators (lines) Δ and vertices V:

 $= <\varphi_i \varphi_j>_0 \equiv \Delta_{ij}^{\varphi\varphi}$

 $= <\varphi_i \varphi'_j>_0 \equiv \Delta_{ij}^{\varphi\varphi'}$

 $= <v'_i v'_j> \equiv \Delta_{ij}^{\varphi'\varphi'} = 0\,,$



$\equiv V_{ijs}.$

The explicit form of propagators can be obtained from the quadratic part of action (35) and in the frequency-wave-vector and time-wave-vector representation have the form:

$$\Delta_{ij}^{\varphi\varphi}(\boldsymbol{k},\omega_k) = \frac{P_{ij}D(k)}{(i\omega_k+\nu_0k^2)(-i\omega_k+\nu_0k^2)},$$
$$\Delta_{ij}^{\varphi\varphi}(\boldsymbol{k},t'-t) = \frac{P_{ij}D(k)}{2\nu_0k^2}e^{-\nu_0k^2|t'-t|},$$
$$\Delta_{ij}^{\varphi\varphi'}(\boldsymbol{k},\omega_k) = \frac{P_{ij}}{(-i\omega_k+\nu_0k^2)},$$
$$\Delta_{ij}^{\varphi\varphi'}(\boldsymbol{k},t'-t) = \theta\,(t'-t)\,e^{-\nu_0k^2(t'-t)}P_{ij} \qquad (35)$$

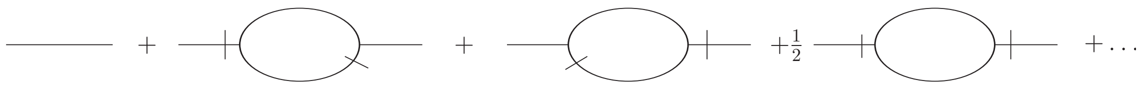
where $P_{ij}=\delta_{ij}-k_ik_j/k^2$ is transverse projector due to incompressibility. Here the step function θ reflects an important physical feature of propagator $\Delta^{\varphi\varphi'}$ namely, its retardation, because actually it is the lowest order approximation of the response function $\langle\varphi\varphi'\rangle$ of the original model (??)–(??). The propagator $\Delta^{\varphi\varphi}$ represents the leading approximation of the pair correlation function of the velocity field $W_{2ij}=\langle\varphi_i\,\varphi_j\rangle$, which in the wave-vector representation is proportional to the kinetic energy spectrum $E(k)$. As it is well known $E(k)$ plays very important role in the equation of energy balance, which describes the cascade of the kinetic energy from the largest scales to the smallest ones, where it dissipates.

The vertex V is given by the non-linear part of (35) and in the frequency-wave-vector representation has the form:

$$V_{ijs}^{k} = i(k_j \delta_{is} + k_s \delta_{ij}), \qquad (36)$$

where $\boldsymbol{k}$ denotes the wave-vector transferred by field φ_i'. At the end of this section as illustration we present the diagrammatic representation of the pair correlation function $W_2 = \langle \varphi \, \varphi \rangle$ in the one loop approximation (first order in coupling constant g_0):

Pair correlation function of velocity field with one-loop precision



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Ward identities

The Ward identities are various relations following from an exact or approximate symmetry of the action. The simplest is the statement that the Green functions are invariant, and it is conveniently stated in the language of the corresponding generating functionals. In the theory of stochastic turbulence we use it to derive the relations between Green functions and composite operators, which allow us to make conclusions about their renormalization.

Useful information about the renormalization of composite operators can also be obtained using the Ward identities for Galilean transformations of fields $\phi \equiv \varphi', varphi$, $\phi \to \phi_v$:

$$\varphi_v(x) = \varphi(x_v) - v(t),\ \varphi'_v(x) = \varphi'(x_v) \tag{37}$$

$$x \equiv \boldsymbol{x}, t;\ x_v \equiv \boldsymbol{x} + u(t), t;\ u(t) = \int_{-\infty}^{t} dt' v(t') = \int_{-\infty}^{\infty} dt' \theta(t-t') v(t'), \tag{38}$$

where $v(t)$ is parameter of transformation. It is arbitrary velocity (vector function) depending only on time and falls well-enough as $|t| \to \infty$.

transformation for action:

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$$S(\phi_v) = S(\phi) + \varphi' \partial_t v \tag{39}$$

Galilean invariance for turbulence reads:

$$G(A) = G(A_v) \Rightarrow \delta_v G(A) = 0$$

functional integral measure is invariant: $D\phi = D\phi_v$

$$\int D\phi \delta_v e^{S(\phi_v)+A\phi_v+aF_v} = 0 \tag{40}$$

$$\int D\phi \left[\phi' \partial_t v + A\partial\phi + a\partial F\right] e^{S(\phi)+A\phi+aF} = 0 \tag{41}$$

$$\partial_v \phi(x) = u\partial\varphi(x) - v,\ \partial_v \phi'(x) = u\partial\varphi'(x), \tag{42}$$

$$\partial_v \partial_t \phi(x) = u\partial\partial_t\varphi(x) + v\partial\varphi(x) - \partial_t v,\ \partial \equiv \nabla \equiv \frac{\partial}{\partial \boldsymbol{x}} \tag{43}$$

$a = 0 \Rightarrow$ Ward identities for Green fuctions

terms linear in $a \Rightarrow$ Ward identities for composite operators

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Ward identities for Green functions

$$\langle\langle \phi' \partial_t v + A \partial \phi \rangle\rangle = 0 \tag{44}$$

(38) $\Rightarrow$

$$\int dt \int d\boldsymbol{x} v(t) \left\langle\left\langle -\partial_t \phi'(x) + \int_{-\infty}^{t} A(\boldsymbol{x}, t') \frac{\partial \phi(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - A_{\varphi}(x) \right\rangle\right\rangle = 0 \tag{45}$$

v is arbitrary $\Rightarrow$

$$\int d\boldsymbol{x} \left\langle\left\langle -\partial_t \phi'(x) + \int_{-\infty}^{\infty} \theta(t - t') A(\boldsymbol{x}, t') \frac{\partial \phi(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - A_{\varphi}(x) \right\rangle\right\rangle = 0$$

$$\phi \quad \text{in} \quad \langle\langle \quad \rangle\rangle \Leftrightarrow \frac{\delta}{\delta A}$$

$$\int d\boldsymbol{x} \left\langle\left\langle -\partial_t \frac{\delta}{\delta A_{\varphi'}(x)} + \int_{-\infty}^{\infty} \theta(t - t') A(\boldsymbol{x}, t') \frac{\partial}{\partial \boldsymbol{x}} \frac{\delta}{\delta A(\boldsymbol{x}, t')} - A_{\varphi}(x) \right\rangle\right\rangle = 0$$

$$G = e^W$$

$$\int d\boldsymbol{x} \left[-\partial_t \frac{\delta W}{\delta A_{\varphi'}(x)} + \int_{-\infty}^{\infty} \theta(t-t') A(\boldsymbol{x}, t') \frac{\partial}{\partial \boldsymbol{x}} \frac{\delta W}{\delta A(\boldsymbol{x}, t')} - A_\varphi(x) \right] = 0 \tag{46}$$

in term of generating functional of one-particle irreducible functions:

$$\Gamma(\alpha) = W(A) - \alpha A, \quad \alpha(x) = \frac{\delta W(A)}{\delta A(x)}, \quad A(x) = -\frac{\delta \Gamma(\alpha)}{\delta \alpha(x)}$$

$$\int d\boldsymbol{x} \left[-\partial_t \alpha_{\varphi'}(x) + \int_{-\infty}^{\infty} \theta(t-t') \frac{\delta \Gamma(\alpha)}{\delta \alpha(\boldsymbol{x}, t')} \frac{\partial \alpha(\boldsymbol{x}, t')}{\partial \boldsymbol{x}} - \frac{\delta \Gamma(\alpha)}{\delta \alpha_\varphi(x)} \right] = 0 \tag{47}$$

$$\Gamma(\alpha) = \alpha_{\varphi'} \Gamma_{\varphi'\varphi} \alpha_\varphi + \frac{1}{2} \alpha_{\varphi'} \Gamma_{\varphi'\varphi\varphi} \alpha_\varphi^2 + \dots \tag{48}$$

$$\alpha_{\varphi'} \Gamma_{\varphi'\varphi\varphi} \alpha_\varphi^2 \equiv \int \int \int dx_1 dx_2 dx_3 \alpha_{\varphi'_i}(x_1) \Gamma_{\varphi'_i \varphi_s \varphi_l}(x_1, x_2, x_3) \alpha_{\varphi_s}(x_2) \alpha_{\varphi_l}(x_3)$$

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$$\Gamma_{\varphi_i'\varphi_s}(x_1,x_2) \equiv \Gamma_{is}(x_1,x_2)\Gamma_{\varphi_i'\varphi_s\varphi_l}(x_1,x_2,x_3) \equiv \Gamma_{isl}(x_1,x_2,x_3)$$

Substitution this expansion to the (47) gives

$$\int d\boldsymbol{x}\ \Gamma_{isl}(x_1,x_2,x) + \left[\theta(t-t_1)\frac{\partial}{\partial x_{1l}} + \theta(t-t_2)\frac{\partial}{\partial x_{2s}}\right]\Gamma_{is}(x_1,x_2) = 0 \tag{49}$$

integration over t: important!!! due to translation invariance of $\Gamma_{is}(x_1,x_2)$ $\frac{\partial}{\partial \boldsymbol{x}_2} = -\frac{\partial}{\partial \boldsymbol{x}_1}$ and from it we obtain in (49)

$$\theta(t-t_1) - \theta(t-t_2) \Rightarrow$$

after integration (49) over t we obtain final expression in time-coordinate reprezentation

$$\int dx\ \Gamma_{isl}(x_1,x_2,x) + (t_2-t_1)\frac{\partial\Gamma_{is}(x_1,x_2)}{\partial x_{1l}} = 0 \tag{50}$$

Ward identity in wave number - frequency representation $p \equiv \boldsymbol{k}, \omega$:

$$\Gamma_{is}(x_1, x_2) = \frac{1}{(2\pi)^{2+d}} \int dp\ \Gamma_{is}(p) e^{ip(x_1 - x_2)}$$

$$\Gamma_{isl}(x_1, x_2, x_3) =$$

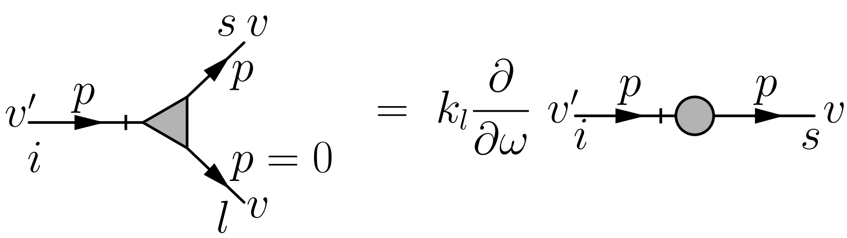
$$\frac{1}{(2\pi)^{3(2+d)}} \int\int\int dp_1 dp_2 dp_3 \delta(p_1 + p_2 + p_3) \Gamma_{isl}(p_1, p_2, p_3) e^{i(p_1 x_1 + p_2 x_2 + p_3 x_3)}$$

$$\int dx\ \Gamma_{isl}(x_1, x_2, x) = \frac{1}{(2\pi)^{(2+d)}} \int dp\ \Gamma_{isl}(p, p, 0) e^{ip(x_1 - x_2)}$$

Finally

$$\Gamma_{isl}(p, p, 0) = k_l \frac{\partial}{\partial \omega} \Gamma_{is}(p) \qquad (51)$$

Graphic representation:



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Conservation laws of energy – momentum in stochastic hydrodynamics

Conservation laws of energy – momentum: important role to understand the processes of redistribution and transfer of energy end momenta in turbulent flow
Stochastic hydrodynamics: energy, momentum, their flows – random quantities constructed on velocity and its derivatives
In field theory: composite operators
Swinger equations $\Rightarrow$ Conservation laws

$$\int D\phi \frac{\delta}{\delta\phi} e^{S(\phi)+A\phi} = 0, \quad \phi \equiv \varphi', \varphi \tag{52}$$

$\Rightarrow$ Composite operator (random quantity) inside of brackets equals to zero

$$\frac{\delta S(\phi)}{\delta\phi} + A(x) = 0 \tag{53}$$

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First Swinger equation in field model of stochastic hydrodynamics:

$$\int D\phi \frac{\delta}{\delta\varphi_i'(x)} e^{S(\phi)+A\phi} = 0 \tag{54}$$

$$A_i^{\varphi'} + D_{is}\varphi_s' - \partial_t\varphi_i + \nu_0\Delta\varphi_i + (\varphi\partial)\varphi_i - \partial_i p = 0 \tag{55}$$

non-local composite operator (pressure):

$$p = -\frac{\partial_l\partial_s}{\Delta}(\varphi_l\varphi_s) \tag{56}$$

Second Swinger equation in field model of stochastic hydrodynamics:

$$\int D\phi \frac{\delta}{\delta\varphi_i'(x)} \varphi_i(x) e^{S(\phi)+A\phi} = 0 \tag{57}$$

$$\varphi A^{\varphi'} + \varphi D\varphi' - \varphi\partial_t\varphi + \varphi\nu_0\Delta\varphi + \varphi(\varphi\partial)\varphi - (\varphi\partial)p = 0 \tag{58}$$

These relations – equations of composite operators: conservation laws of energy and momentum

$$\partial_t \varphi_i + \partial_k \Pi_{ik} = D_{ik} \varphi'_k + A_i^{\varphi'} \quad (59)$$

$$\partial_t E + \partial_i S_i = \varepsilon + \varphi D \varphi' + \varphi A^{\varphi'} \quad (60)$$

conservation laws for densities of conserved quatities (per unit mass, $\rho = 0$):
φ_i – momentum density, $E = \varphi^2/2$ – energy density

tensor of momentum flow density:

$$\Pi_{ik} = p\delta_{ik} + \varphi_i \varphi_k - \nu_0 (\partial_i \varphi_k + \partial_k \varphi_i) \quad (61)$$

vector of energy flow density:

$$S_i = p\varphi_i - \nu_0 \varphi_k (\partial_i \varphi_k + \partial_k \varphi_i) + \frac{1}{2} \varphi_i \varphi^2 \quad (62)$$

energy dissipation rate: tensor of momentum flow density:

$$\varepsilon = \frac{1}{2} \nu_0 (\partial_i \varphi_k + \partial_k \varphi_i)^2 \quad (63)$$

Averaging equations over ϕ with weight $expS(\phi)$ $\Rightarrow$ energy momentum balance equations

energy balance equation at vanishing external non-random forcing $A^{\varphi'}$:

$$\partial_t \langle E \rangle + \partial_i \langle S_i \rangle = - \langle \varepsilon \rangle + \langle \varphi D \varphi' \rangle \tag{64}$$

homogeneous an isotropic theory at vanishing external forcing $\Rightarrow$ mean value $\langle F(x) \rangle$ of arbitrary composite operator $F(x)$ independent of x i.e. constant $\Rightarrow$ all its derivatives vanish:

$$0 = -\bar{\varepsilon} + \int \int \boldsymbol{x}' dt' \langle \varphi_i(x) \varphi'_s(x') \rangle D_{is}(x, x'),\ \bar{\varepsilon} = \langle \varepsilon \rangle \tag{65}$$

$$D_{ij}(x, x') = \frac{\delta(t - t')}{(2\pi)^d} \int \mathrm{d}\boldsymbol{k}\, D(k) P_{ij}(\boldsymbol{k}) e^{i\boldsymbol{k}(\boldsymbol{x} - \boldsymbol{x}')} \equiv \delta(t - t') d_{is}(\boldsymbol{x}, \boldsymbol{x}') \tag{66}$$

$$\bar{\varepsilon} = \int \boldsymbol{x}' \langle \varphi_i(x) \varphi'_s(x') \rangle |_{t=t'} d_{is}(\boldsymbol{x}, \boldsymbol{x}') \tag{67}$$

$$\langle \varphi_i(x) \varphi'_s(x') \rangle |_{t=t'} = \frac{1}{2} \delta(\boldsymbol{x} - \boldsymbol{x}') P_{is} \tag{68}$$

Stationary homogeneous isotropic developed turbulence

pair correlator of random force f is expressed via the energy dissipation rate ($=$ pumping power):

$$\bar{\varepsilon} = \frac{1}{2} tr\, d(\boldsymbol{x}, \boldsymbol{x}) \tag{69}$$

or

$$\bar{\varepsilon} = \frac{d-1}{2(2\pi)^d} \int \mathrm{d}\boldsymbol{k}\, d(k), \quad P_{ii}(\boldsymbol{k}) = tr P(\boldsymbol{k}) = d-1, \quad k \equiv \boldsymbol{k} \tag{70}$$

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